# *Finite Element Formulations for Near-Incompressibility*

## The near-incompressibility problem:

- Standard low order displacement-based finite elements perform very poorly in the solution solid mechanics problems with nearly incompressible materials (von Mises elasto-plasticity is a typical example).
- These elements ***lock***, i.e., produce overstiff solutions with unacceptable error.
- Low order elements are preferred in the solution of complex problems due to their simplicity.
- But can only be used under near-incompressibility if properly combined with special techniques.

## The F-bar method. Basic idea

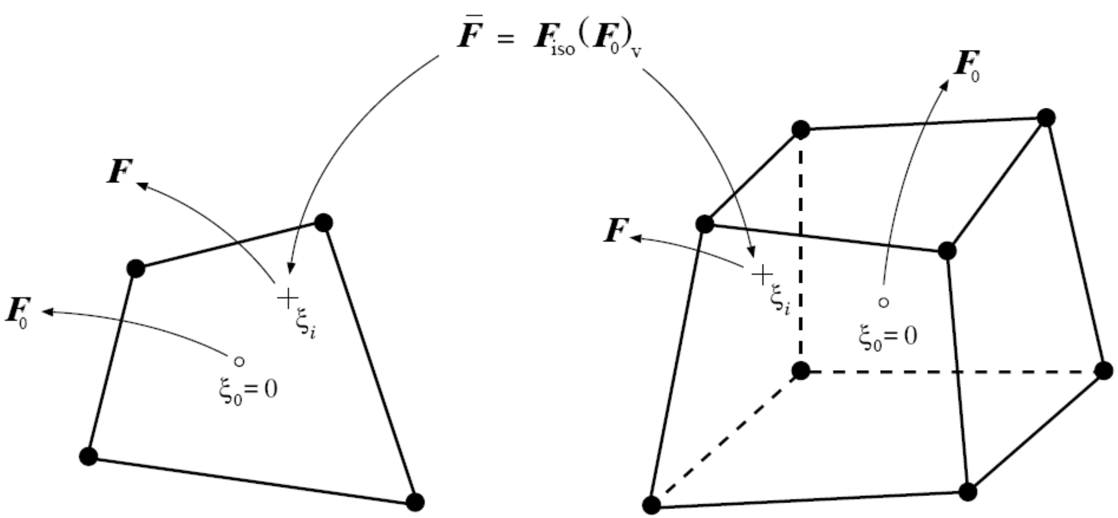
- Conventional elements impose near-incompressibility at each Gauss point

- This overconstrains the element if the underlying material is nearly incompressible

- The F-bar method is based on the idea of ***relaxing the volumetric constraint***

- How ? By using a form of averaged volumetric strain over the element.

# The elements




Deviatoric/volumetric split of $\boldsymbol{F}$

$$\boldsymbol{F}_{\text{iso}} = (\det \boldsymbol{F})^{-1/3}\, \boldsymbol{F}$$

$$\boldsymbol{F}_{\text{v}} = (\det \boldsymbol{F})^{1/3}\, \boldsymbol{I}$$

The F-bar deformation gradient

$$\bar{\boldsymbol{F}} = \boldsymbol{F}_{\text{iso}}\,(\boldsymbol{F}_0)_{\text{v}} = \left(\frac{\det \boldsymbol{F}_0}{\det \boldsymbol{F}}\right)^{1/3} \boldsymbol{F}.$$

Gauss point stress

$$\boldsymbol{\sigma}_{n+1} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \bar{\boldsymbol{F}})$$

Element internal force vector

$$\mathbf{f}_{(e)}^{\text{int}} = \int_{\varphi(\Omega^{(e)})} \mathbf{B}^T \, \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \bar{\boldsymbol{F}}) \, \mathrm{d}v$$

Tangent stiffness matrix

$$\mathbf{K}_T^{(e)} = \underbrace{\int_{\varphi(\Omega^{(e)})} \mathbf{G}^T \; \mathsf{a}\big|_{\boldsymbol{F}=\bar{\boldsymbol{F}}} \, \mathbf{G} \, \mathrm{d}v}_{\substack{\text{standard element} \\ \text{stiffness at } \boldsymbol{F}=\bar{\boldsymbol{F}}}} + \underbrace{\int_{\varphi(\Omega^{(e)})} \mathbf{G}^T \, \mathsf{q} \, (\mathbf{G}_0 - \mathbf{G}) \, \mathrm{d}v}_{\substack{\text{additional} \\ \text{stiffness}}}$$

$$\mathsf{q} = \tfrac{1}{3}\,\mathsf{a} : (\boldsymbol{I} \otimes \boldsymbol{I}) - \tfrac{2}{3}(\boldsymbol{\sigma} \otimes \boldsymbol{I})$$

Easy incorporation into existing codes with conventional elements !

# Internal force computation

(i)
- compute $\mathbf{G}_0$ (standard $\mathbf{G}$ matrix at $\boldsymbol{\xi} = \boldsymbol{\xi}_0$)
- evaluate $\det \boldsymbol{F}_0$, with $\boldsymbol{F}_0 := \boldsymbol{I} + \mathbf{G}_0\, \mathbf{u}$

(ii) set Gauss point coordinates $\boldsymbol{\xi}_i$, weights $w_i$ and Jacobian determinants $j_i$

(iii) do $i = 1, n_{\text{gausp}}$ (loop over Gauss points)

- compute standard $\mathbf{G}$ matrix at $\boldsymbol{\xi}_i$
- $\boldsymbol{F} := \boldsymbol{I} + \mathbf{G}\, \mathbf{u}$ (conventional def. gradient)
- $\bar{\boldsymbol{F}} := \left( \dfrac{\det \boldsymbol{F}_0}{\det \boldsymbol{F}} \right)^{\frac{1}{3}} \boldsymbol{F}$ (modified def. gradient)
- $\boldsymbol{\sigma} := \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \bar{\boldsymbol{F}})$ (call stress update routine)
- compute standard $\mathbf{B}$ matrix at $\boldsymbol{\xi}_i$
- $\mathbf{f}^{\text{int}}_{(e)} := \mathbf{f}^{\text{int}}_{(e)} + w_i\, j_i\, \mathbf{B}^T \boldsymbol{\sigma}$ (add $i^{\text{th}}$ Gauss point contrib.)

end do

## Stiffness matrix computation

(i)
- compute $\mathbf{G}_0$ (standard $\mathbf{G}$ matrix at $\boldsymbol{\xi}=\boldsymbol{\xi}_0$)
- evaluate $\det \boldsymbol{F}_0$, with $\boldsymbol{F}_0 := \boldsymbol{I} + \mathbf{G}_0\,\mathbf{u}$

(ii) set Gauss point coordinates $\boldsymbol{\xi}_i$, weights $w_i$ and Jacobian determinants $j_i$

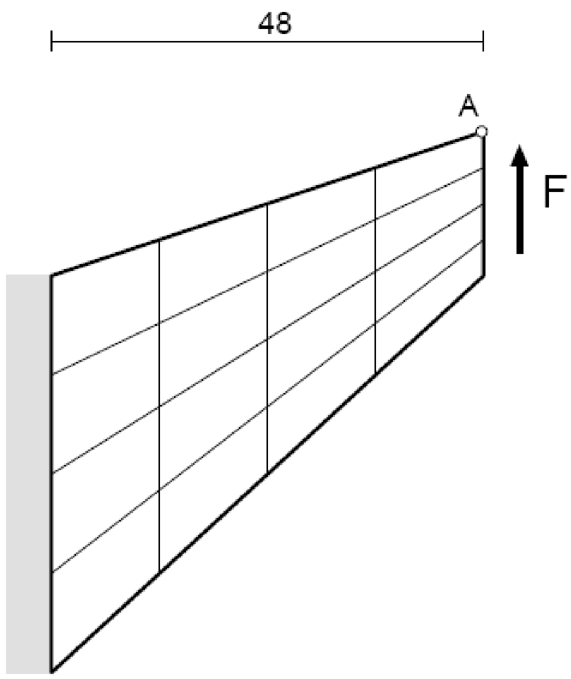
(iii) do $i = 1, n_{\text{gausp}}$ (loop over Gauss points)

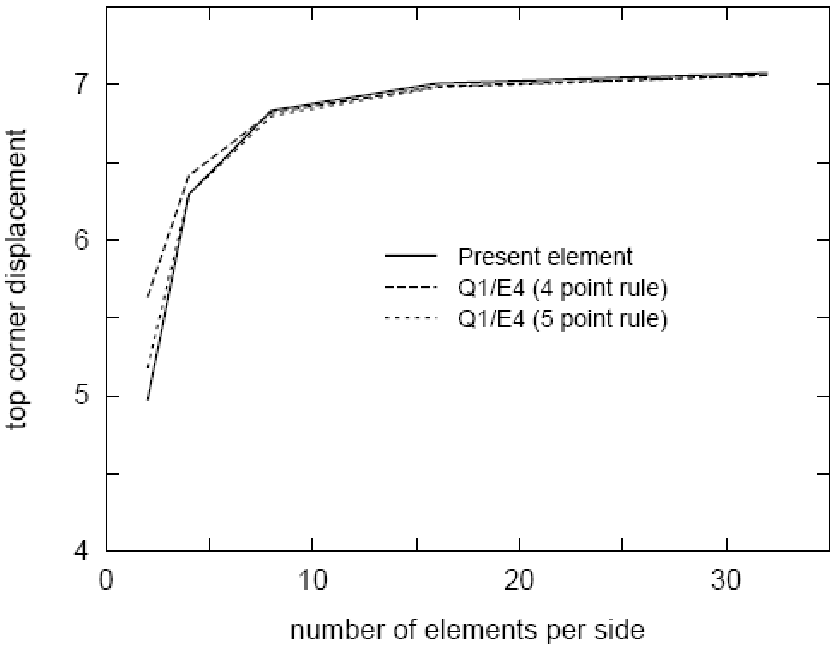
- compute standard $\mathbf{G}$ matrix at $\boldsymbol{\xi}_i$
- $\boldsymbol{F} := \boldsymbol{I} + \mathbf{G}\,\mathbf{u}$
- $\bar{\boldsymbol{F}} := \left(\dfrac{\det \boldsymbol{F}_0}{\det \boldsymbol{F}}\right)^{1/3} \boldsymbol{F}$ (modified def. gradient)
- $\mathbf{a} := \hat{\mathbf{a}}(\bar{\boldsymbol{F}})$ (tangent modulus computation routine)
- $\mathbf{K}_T^{(e)} := \mathbf{K}_T^{(e)} + w_i\, j_i\, \mathbf{G}^T \mathbf{a}\, \mathbf{G}$
- 
  – compute matrix $\mathbf{q}$ defined by (15.11)

  – $\mathbf{K}_T^{(e)} := \mathbf{K}_T^{(e)} + w_i\, j_i\, \mathbf{G}^T \mathbf{q}\, (\mathbf{G}_0 - \mathbf{G})$

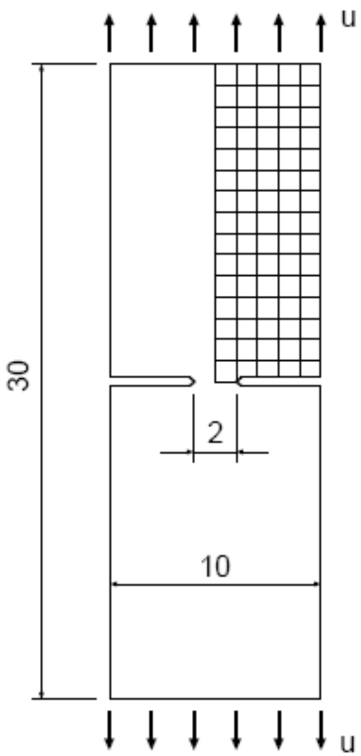
end do

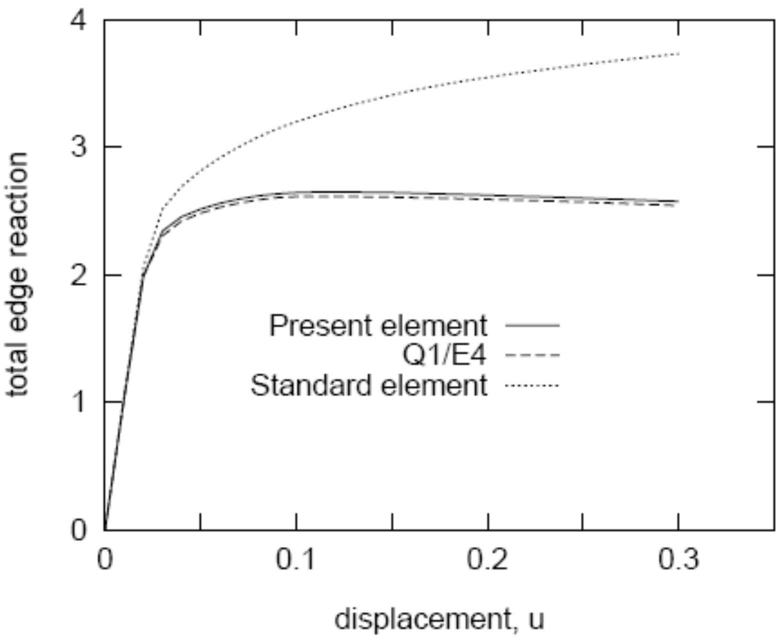
# Verification. Cook's membrane (quasi-incompressible hyperelasticity)



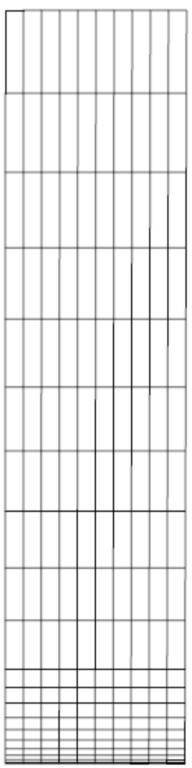



# Verification. Stretching of a double notched specimen (perfect plasticity)

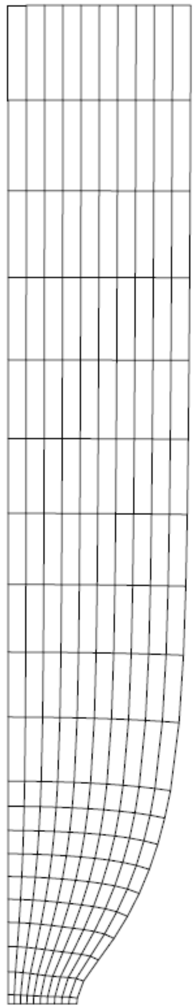


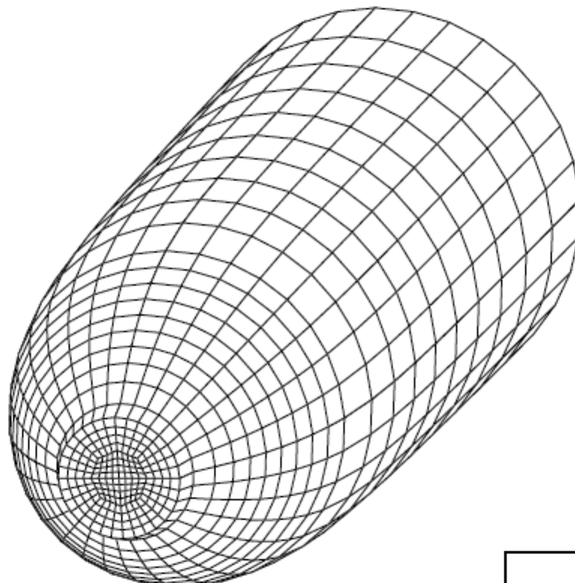
# Verification. Necking of an axisymmetric bar



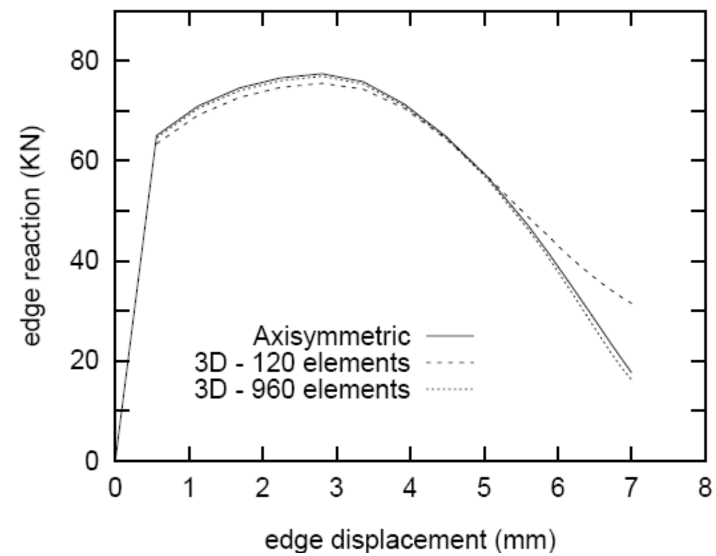
initial mesh



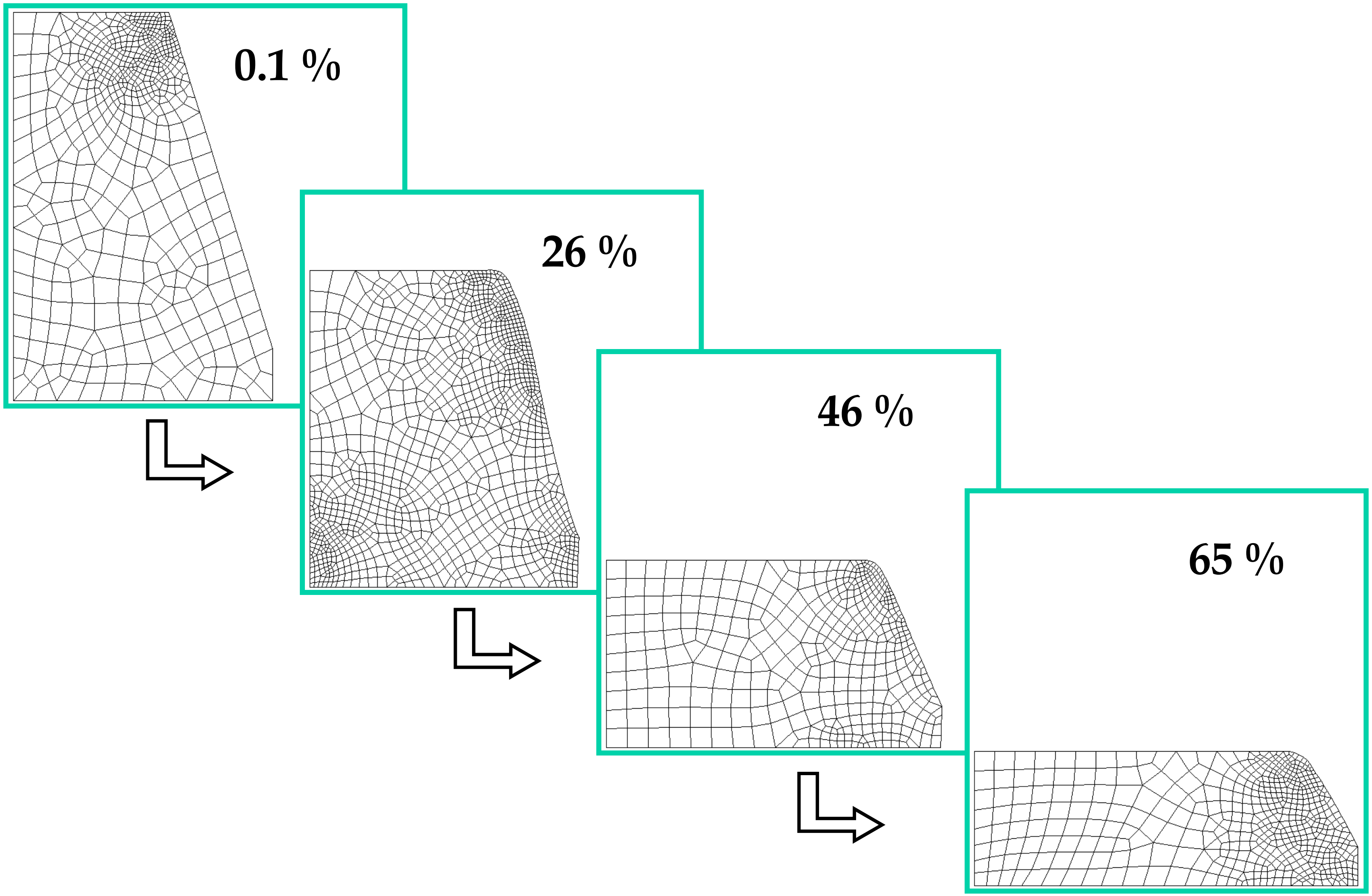
deformed mesh



3-D deformed mesh

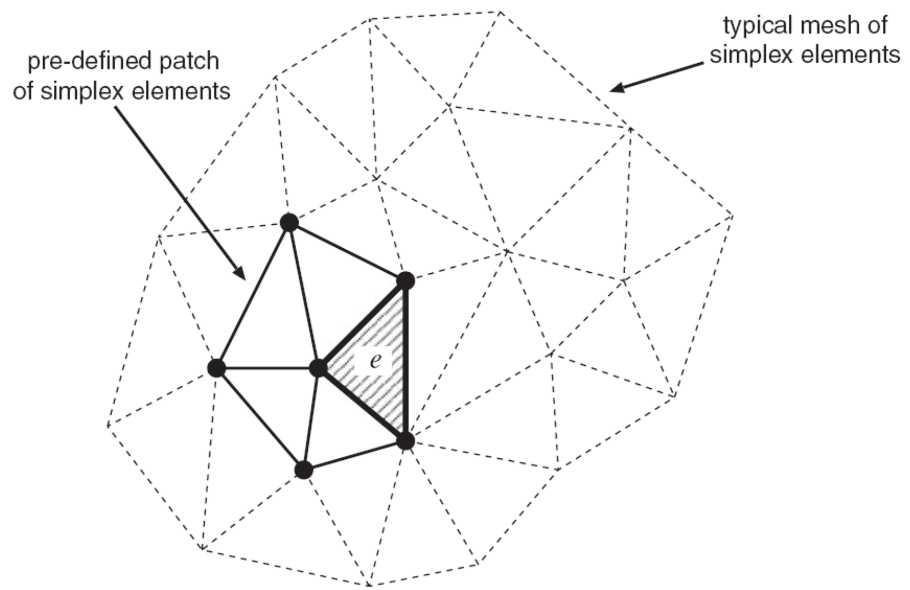


# Application. Upsetting of a tapered cylinder (adaptive remeshing)



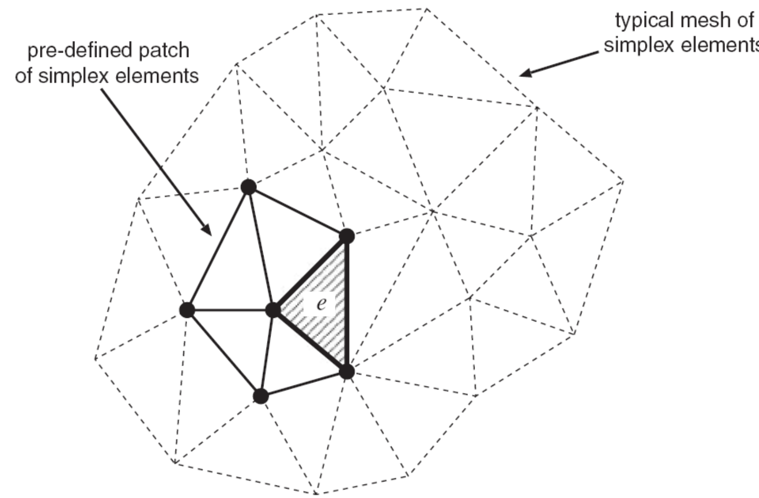
## The F-bar-Patch method for simplex elements

- The conventional F-bar concept cannot be used for simplex elements
- The idea then is to average the volumetric strain over a ***patch*** of elements



## F-bar deformation gradient

$$\bar{\boldsymbol{F}}_e = \left[ \frac{v_{\text{patch}}}{V_{\text{patch}}(\det \boldsymbol{F}_e)} \right]^{1/3} \boldsymbol{F}_e$$

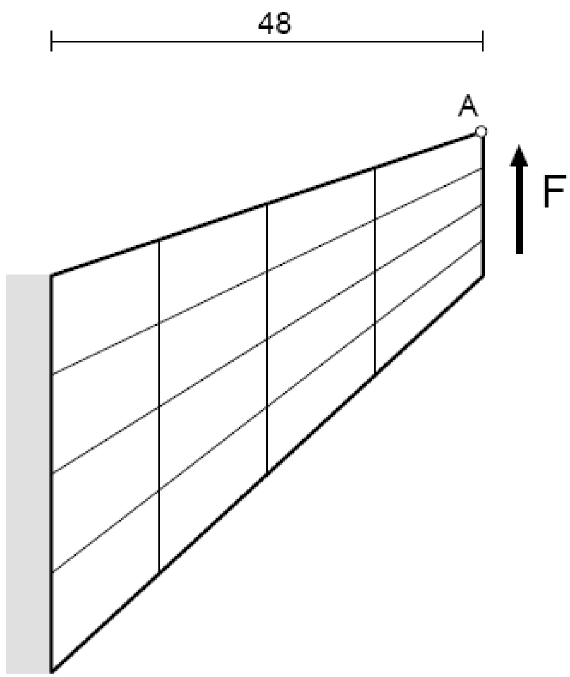


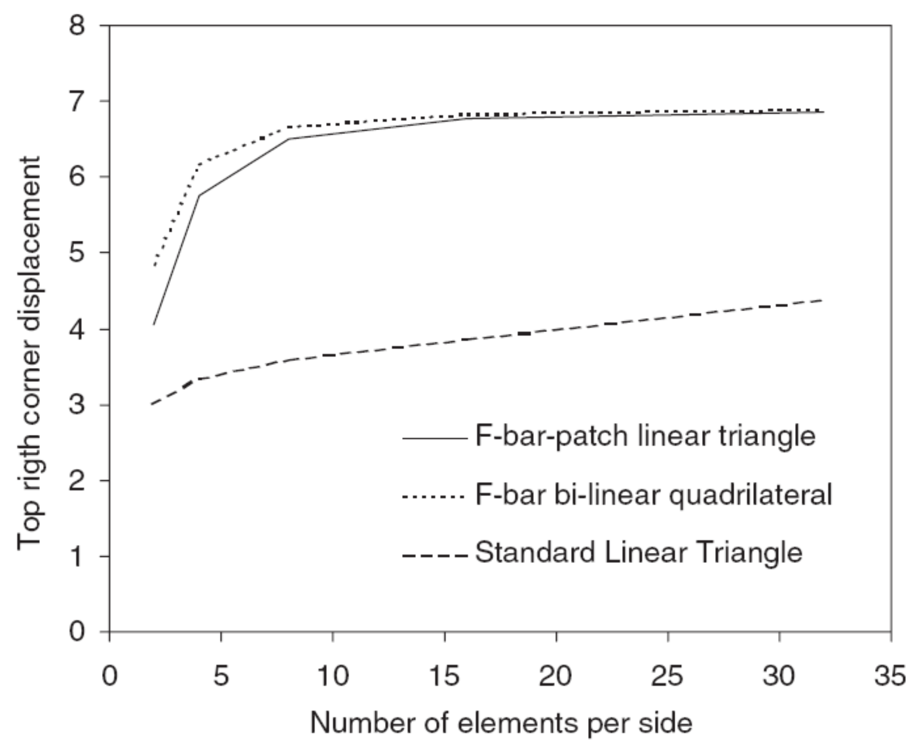

## Tangent stiffness matrix

$$\mathbf{K}_{ee} = \int_{\varphi(\Omega_e)} \mathbf{G}_e^{\mathrm{T}} \mathbf{a} \mathbf{G}_e \, \mathrm{d}v + \left( \frac{v_e}{v_{\text{patch}}} - 1 \right) \int_{\varphi(\Omega_e)} \mathbf{G}_e^{\mathrm{T}} \mathbf{q} \mathbf{G}_e \, \mathrm{d}v$$

$$\mathbf{K}_{es} = \frac{v_e}{v_{\text{patch}}} \int_{\varphi(\Omega_e)} \mathbf{G}_e^{\mathrm{T}} \mathbf{q} \mathbf{G}_s \, \mathrm{d}v, \quad s \in \mathscr{P}; \;\; s \neq e$$

# Verification. Cook's membrane (quasi-incompressible hyperelasticity)



# Verification. Indentation of a rubber block (quasi-incompressible hyperelasticity)

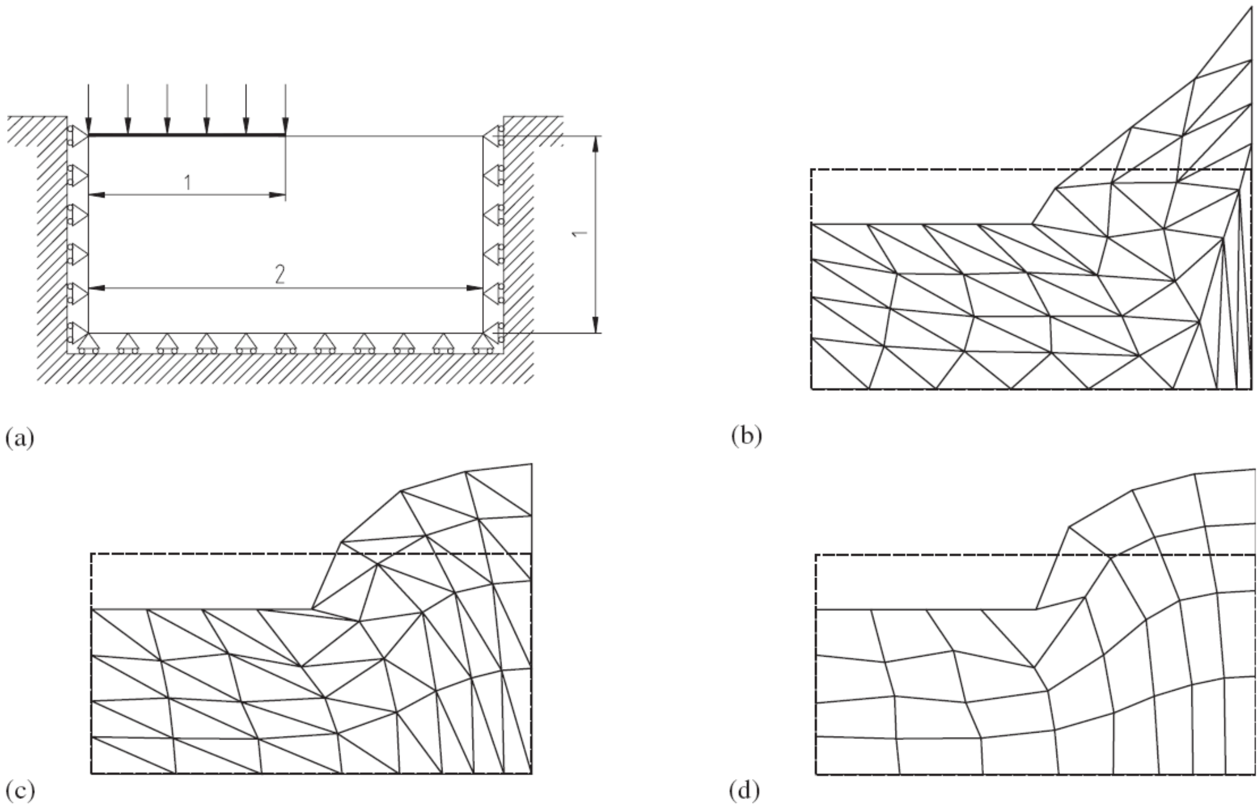



Figure 5. Indentation of a rubber block: (a) problem definition; deformed meshes at 25% compression; (b) standard isoparametric 3-noded triangle; (c) F-bar-Patch 3-noded triangle with patches of two elements; and (d) 4-noded F-bar quadrilateral.

# Verification. Necking of an axisymmetric bar