Plasticity with Singular Yield Surfaces

Topics:

- *Tresca model in principal stress space*
- *Mohr-Coulomb model. Geomaterials*
- *Finite strain single crystal plasticity*

General Return Mapping-based Algorithm for Plasticity.
REVIEW

Problem 7.1 (The elastoplastic constitutive initial value problem). *Given the initial values* $\boldsymbol{\varepsilon}^e(t_0)$ *and* $\boldsymbol{\alpha}(t_0)$ *and given the history of the strain tensor,* $\boldsymbol{\varepsilon}(t)$, $t \in [t_0, T]$, *find the functions* $\boldsymbol{\varepsilon}^e(t)$, $\boldsymbol{\alpha}(t)$ *and* $\dot{\gamma}(t)$ *for the elastic strain, hardening internal variables set and plastic multiplier that satisfy the reduced general elastoplastic constitutive equations*

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^e(t) &= \dot{\boldsymbol{\varepsilon}}(t) - \dot{\gamma}(t)\, \boldsymbol{N}(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) \\ \dot{\boldsymbol{\alpha}}(t) &= \dot{\gamma}(t)\, \boldsymbol{H}(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) \end{aligned} \tag{7.6}$$

$$\dot{\gamma}(t) \geq 0, \quad \Phi(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) \leq 0, \quad \dot{\gamma}(t)\, \Phi(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) = 0 \tag{7.7}$$

for each instant $t \in [t_0, T]$, *with*

$$\boldsymbol{\sigma}(t) = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\bigg|_t, \quad \boldsymbol{A}(t) = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\alpha}}\bigg|_t. \tag{7.8}$$

Problem 7.2 (The incremental elastoplastic constitutive problem). *Given the values* ε^e_n *and* $\boldsymbol{\alpha}_n$*, of the elastic strain and internal variables set at the beginning of the pseudo-time interval* $[t_n, t_{n+1}]$*, and given the prescribed incremental strain* $\Delta\varepsilon$ *for this interval, solve the following system of algebraic equations*

$$\begin{aligned} \varepsilon^e_{n+1} &= \varepsilon^e_n + \Delta\varepsilon - \Delta\gamma\, \boldsymbol{N}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\gamma\, \boldsymbol{H}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \end{aligned} \tag{7.10}$$

for the unknowns ε^e_{n+1}, $\boldsymbol{\alpha}_{n+1}$ *and* $\Delta\gamma$*, subjected to the constraints*

$$\Delta\gamma \geq 0, \quad \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \leq 0, \quad \Delta\gamma\, \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) = 0, \tag{7.11}$$

where

$$\boldsymbol{\sigma}_{n+1} = \bar{\rho}\, \frac{\partial\psi}{\partial\varepsilon^e}\bigg|_{n+1}, \quad \boldsymbol{A}_{n+1} = \bar{\rho}\, \frac{\partial\psi}{\partial\alpha}\bigg|_{n+1}. \tag{7.12}$$

(i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n, evaluate the *elastic trial state*

$$\boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1} = \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}$$

$$\boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} = \boldsymbol{\alpha}_{n}$$

$$\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{e}}\bigg|^{\mathrm{trial}}_{n+1}, \qquad \boldsymbol{A}^{\mathrm{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\bigg|^{\mathrm{trial}}_{n+1}$$

(ii) Check plastic admissibility

$$\text{IF } \Phi(\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}, \boldsymbol{A}^{\mathrm{trial}}_{n+1}) \leq 0$$

$$\text{THEN set} \quad (\cdot)_{n+1} = (\cdot)^{\mathrm{trial}}_{n+1} \quad \text{and EXIT}$$

(iii) Return mapping. Solve the system

$$\begin{Bmatrix} \boldsymbol{\varepsilon}^{e}_{n+1} - \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1} + \Delta\gamma\,\boldsymbol{N}_{n+1} \\ \boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} - \Delta\gamma\,\boldsymbol{H}_{n+1} \\ \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{Bmatrix}$$

for $\boldsymbol{\varepsilon}^{e}_{n+1}$, $\boldsymbol{\alpha}_{n+1}$ and $\Delta\gamma$, with

$$\boldsymbol{\sigma}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{e}}\bigg|_{n+1}, \qquad \boldsymbol{A}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\bigg|_{n+1}$$

(iv) EXIT

… for the von Mises model with isotropic hardening…

(i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n, evaluate the *elastic trial state*

$$\boldsymbol{\varepsilon}^{e\,\text{trial}}_{n+1} := \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}$$

$$\bar{\varepsilon}^{p\,\text{trial}}_{n+1} := \bar{\varepsilon}^{p}_{n}$$

$$p^{\text{trial}}_{n+1} := K\,\varepsilon^{e\,\text{trial}}_{\text{v}\;n+1}; \qquad \boldsymbol{s}^{\text{trial}}_{n+1} := 2G\,\boldsymbol{\varepsilon}^{e\,\text{trial}}_{\text{d}\;n+1}$$

$$q^{\text{trial}}_{n+1} := \sqrt{\tfrac{3}{2}\,\boldsymbol{s}^{\text{trial}}_{n+1} : \boldsymbol{s}^{\text{trial}}_{n+1}}$$

(ii) Check plastic admissibility

$$\text{IF } q^{\text{trial}}_{n+1} - \sigma_y(\bar{\varepsilon}^{p\,\text{trial}}_{n+1}) \leq 0$$

$$\text{THEN set} \quad (\cdot)_{n+1} := (\cdot)^{\text{trial}}_{n+1} \quad \text{and EXIT}$$

(iii) Return mapping. Solve the equation

$$\tilde{\Phi}(\Delta\gamma) \equiv q^{\text{trial}}_{n+1} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}^{p}_{n} + \Delta\gamma) = 0$$

for $\Delta\gamma$ using the Newton–Raphson method – GOTO Box 7.4 – and update the state variables

$$p_{n+1} := p^{\text{trial}}_{n+1}; \qquad \boldsymbol{s}_{n+1} := \left(1 - \frac{\Delta\gamma\,3G}{q^{\text{trial}}_{n+1}}\right)\boldsymbol{s}^{\text{trial}}_{n+1}$$

$$\boldsymbol{\sigma}_{n+1} := \boldsymbol{s}_{n+1} + p_{n+1}\,\boldsymbol{I}$$

$$\boldsymbol{\varepsilon}^{e}_{n+1} = \frac{1}{2G}\,\boldsymbol{s}_{n+1} + \frac{1}{3}\,\varepsilon^{e\,\text{trial}}_{\text{v}\;n+1}\,\boldsymbol{I}$$

$$\bar{\varepsilon}^{p}_{n+1} := \bar{\varepsilon}^{p}_{n} + \Delta\gamma$$

(iv) EXIT

Tresca Model in Principal Stress Space

Subgradients and the subdifferential

Let us consider a *scalar* function $y : \mathscr{R}^n \rightarrow \mathscr{R}$. The *subdifferential* of y at a point $\bar{\boldsymbol{x}}$ is the set

$$\partial y(\bar{\boldsymbol{x}}) = \{ s \in \mathscr{R}^n \mid y(\boldsymbol{x}) - y(\bar{\boldsymbol{x}}) \geq \boldsymbol{s} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}), \ \forall\, \boldsymbol{x} \in \mathscr{R}^n \}. \tag{6.69}$$

If the set ∂y is not empty at $\bar{\boldsymbol{x}}$, the function y is said to be *subdifferentiable* at $\bar{\boldsymbol{x}}$. The elements of ∂y are called *subgradients* of y. If the function y is *differentiable*, then the subdifferential contains a *unique* subgradient which coincides with the derivative of y,

$$\partial y = \left\{ \frac{\mathrm{d}y}{\mathrm{d}\boldsymbol{x}} \right\}. \tag{6.70}$$

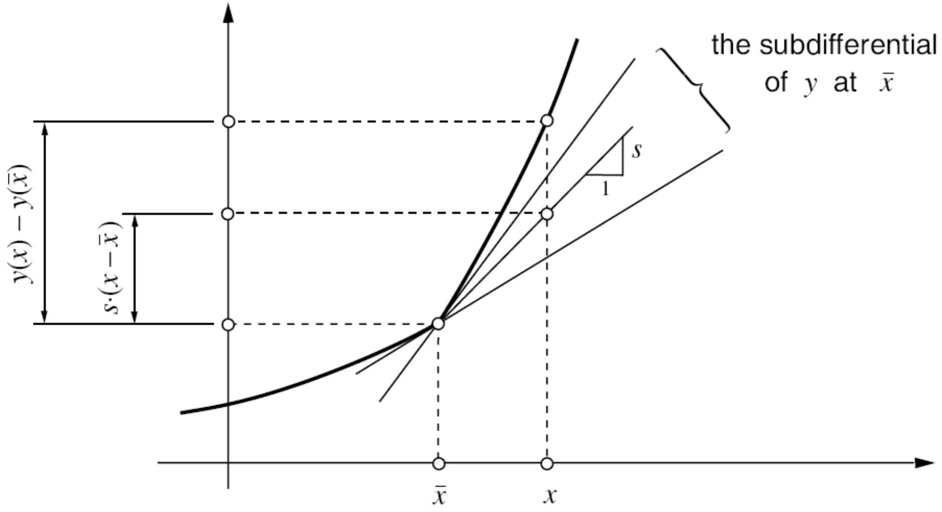


Figure 6.6. The subdifferential of a convex function.

Plastic flow rule for non-smooth (sub-differentiable) potentials

$$N \in \partial_{\sigma} \Psi$$

$$H \in -\partial_{A} \Psi$$

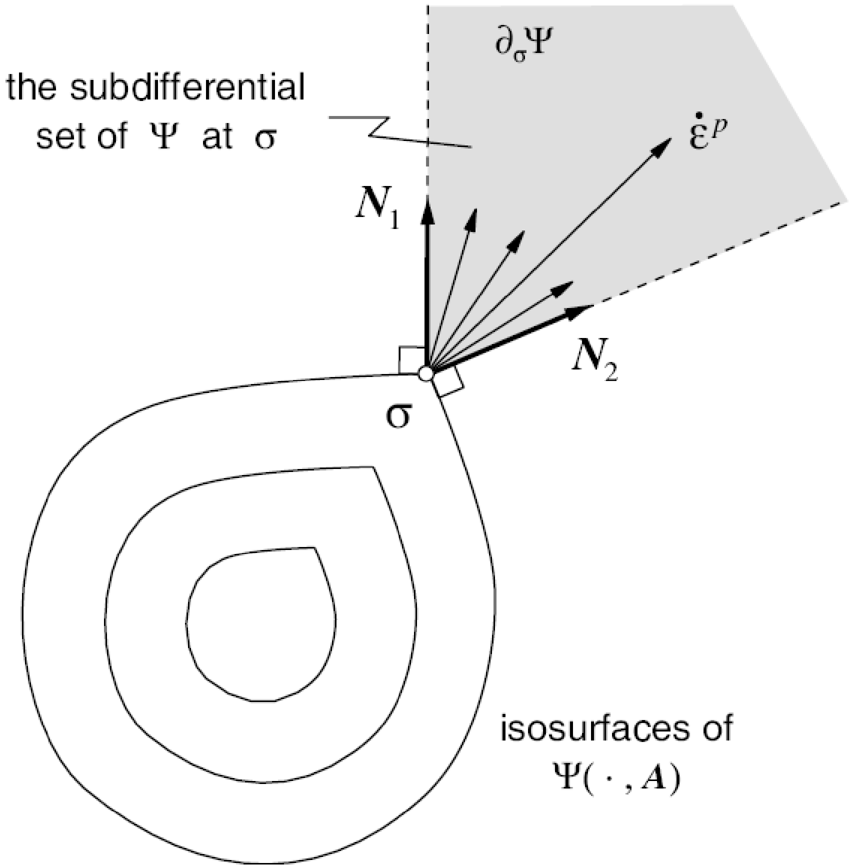


Figure 6.7. The flow vector. Non-smooth potential.

Tresca yield function in principal stress space

$$\Phi(\boldsymbol{\sigma}, \sigma_y) = \sigma_{\max} - \sigma_{\min} - \sigma_y$$

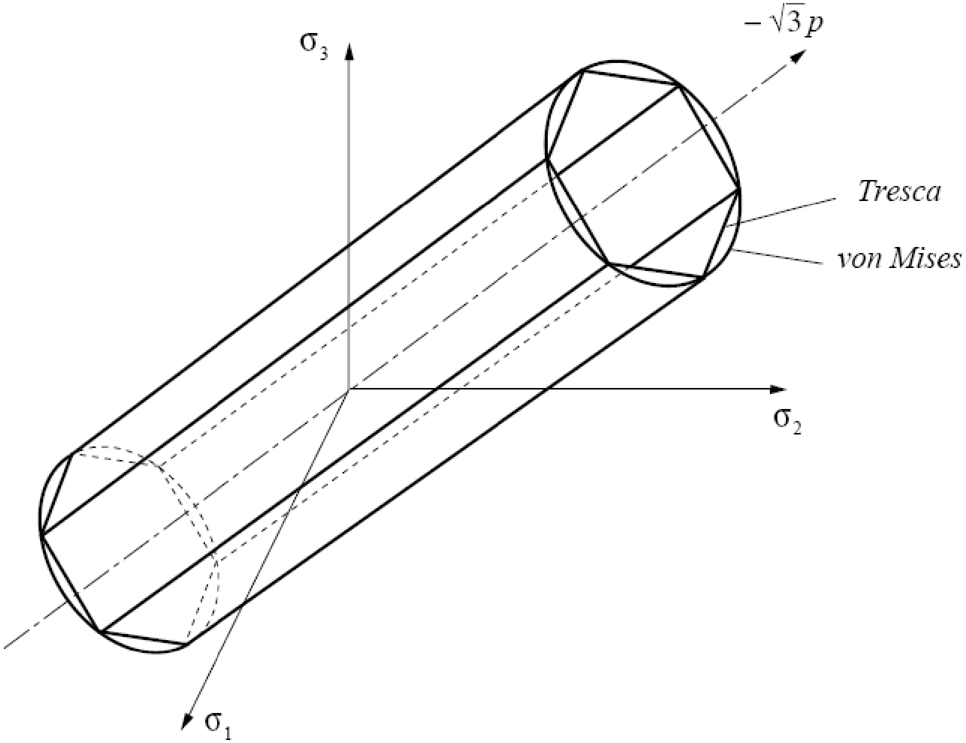


Figure 6.8: The Tresca and von Mises yield surfaces in principal stress space.

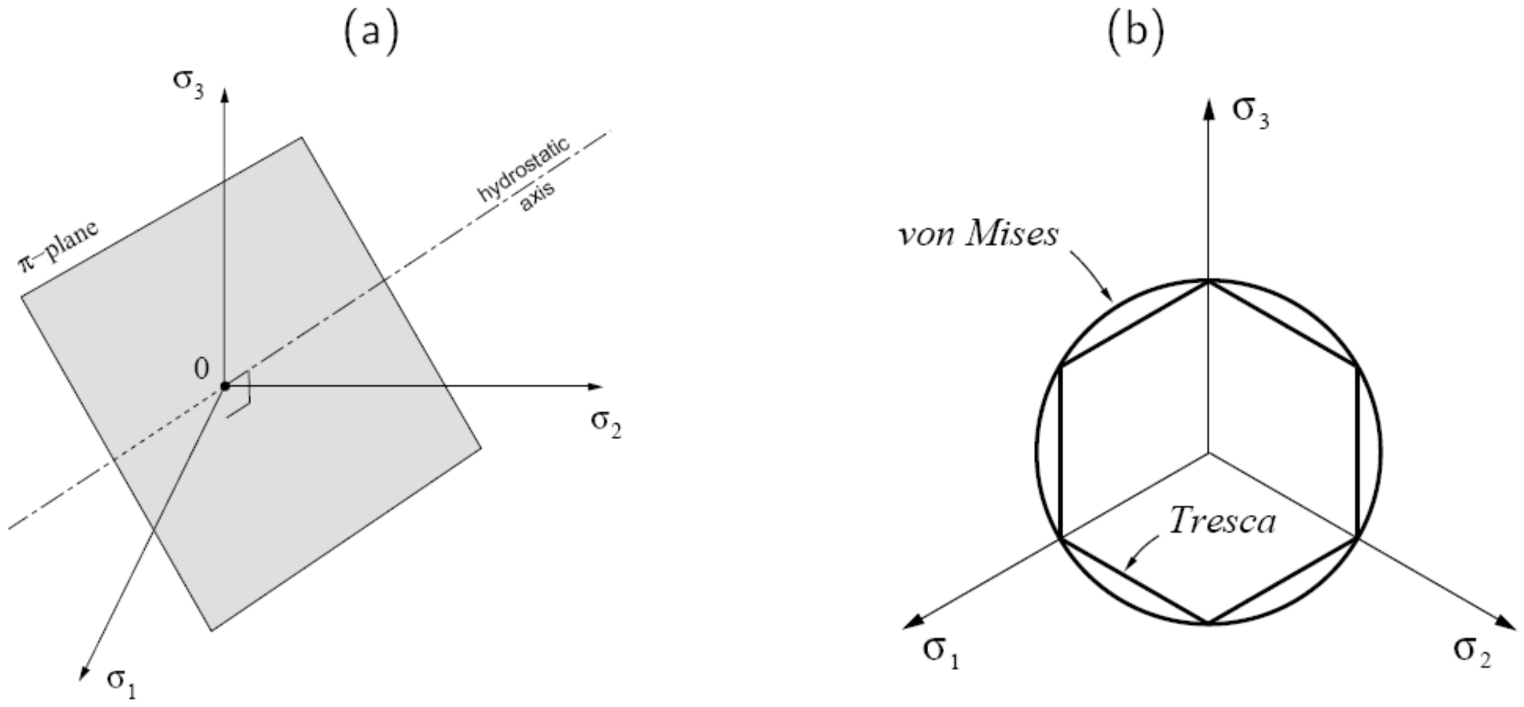


Figure 6.9: (a) The π-plane in principal stress space and, (b) The π-plane representation of the Tresca and von Mises yield surfaces.

Invariant representation

$$\Phi = 2\sqrt{J_2}\cos\theta - \sigma_y$$

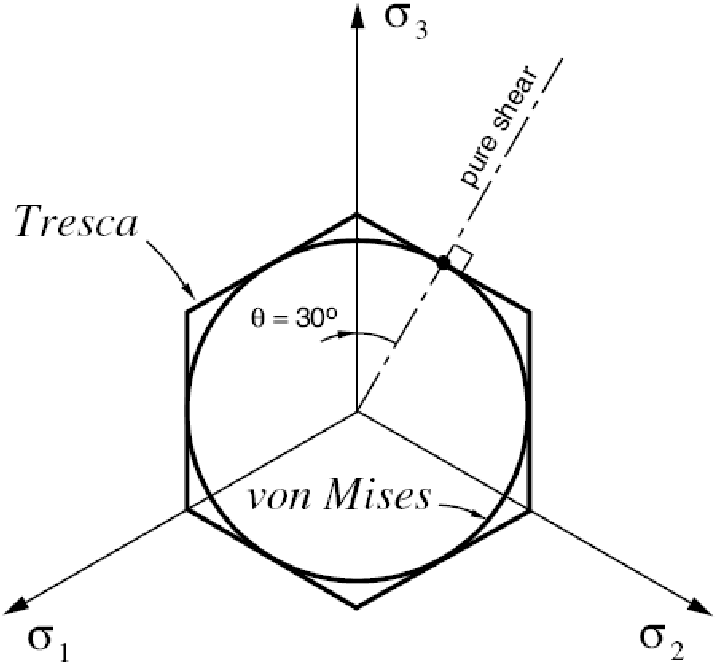
$$J_2 \equiv -I_2(\boldsymbol{s}) = \tfrac{1}{2}\,\mathrm{tr}[\boldsymbol{s}^2] = \tfrac{1}{2}\,\boldsymbol{s}:\boldsymbol{s} = \tfrac{1}{2}\,\|\boldsymbol{s}\|^2$$

$$\boldsymbol{s} \equiv \boldsymbol{\sigma} - \tfrac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\boldsymbol{I}$$

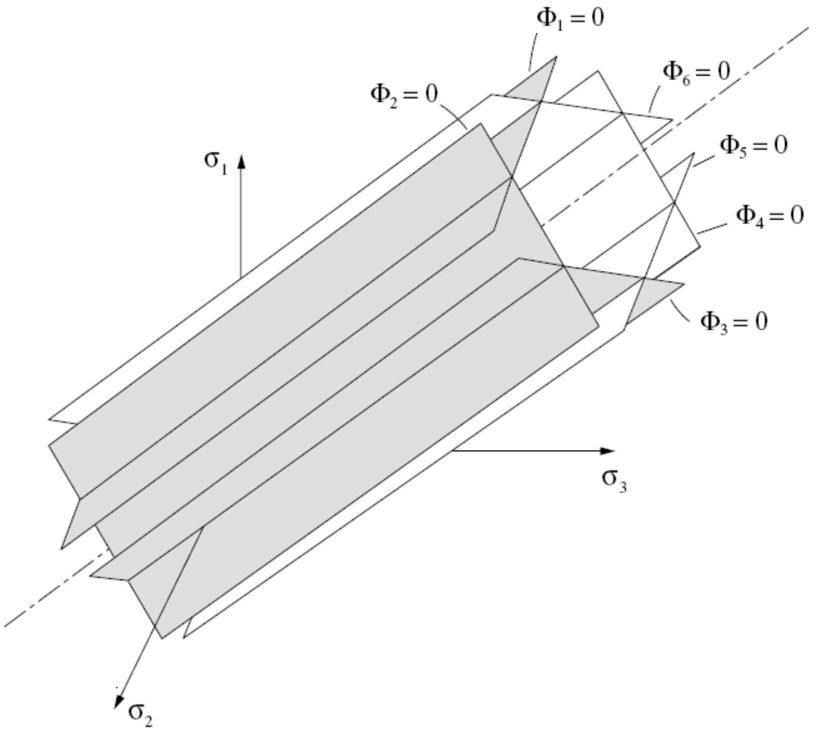
$$\theta \equiv \tfrac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{3}\,J_3}{2J_2^{\frac{3}{2}}}\right)$$

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$

$$J_3 \equiv I_3(\boldsymbol{s}) \equiv \det\boldsymbol{s} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{s})^3$$



Multi-surface representation



Elastic domain

$$\mathscr{E} = \{ \, \boldsymbol{\sigma} \mid \Phi_i(\boldsymbol{\sigma}, \sigma_y) < 0 \quad i = 1, ..., 6 \, \}$$

Yield functions (**linear** on princ. stresses)

$$\Phi_1(\boldsymbol{\sigma}, \sigma_y) = \sigma_1 - \sigma_3 - \sigma_y$$

$$\Phi_2(\boldsymbol{\sigma}, \sigma_y) = \sigma_2 - \sigma_3 - \sigma_y$$

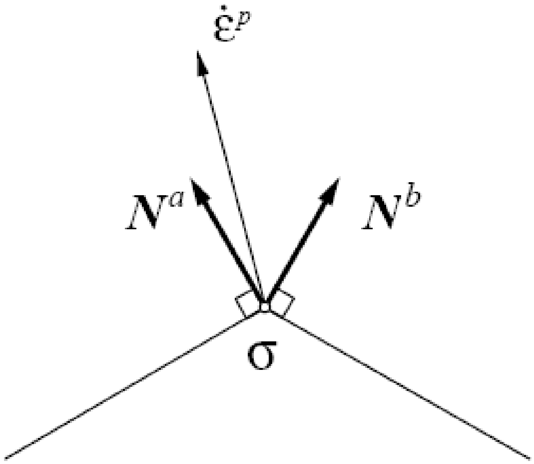
$$\Phi_3(\boldsymbol{\sigma}, \sigma_y) = \sigma_2 - \sigma_1 - \sigma_y$$

$$\Phi_4(\boldsymbol{\sigma}, \sigma_y) = \sigma_3 - \sigma_1 - \sigma_y$$

$$\Phi_5(\boldsymbol{\sigma}, \sigma_y) = \sigma_3 - \sigma_2 - \sigma_y$$

$$\Phi_6(\boldsymbol{\sigma}, \sigma_y) = \sigma_1 - \sigma_2 - \sigma_y,$$

Associative Tresca flow rule



General (subdifferential) form

$$\dot{\varepsilon}^p = \dot{\gamma} \boldsymbol{N}$$

$$\boldsymbol{N} \in \partial_\sigma \Phi$$

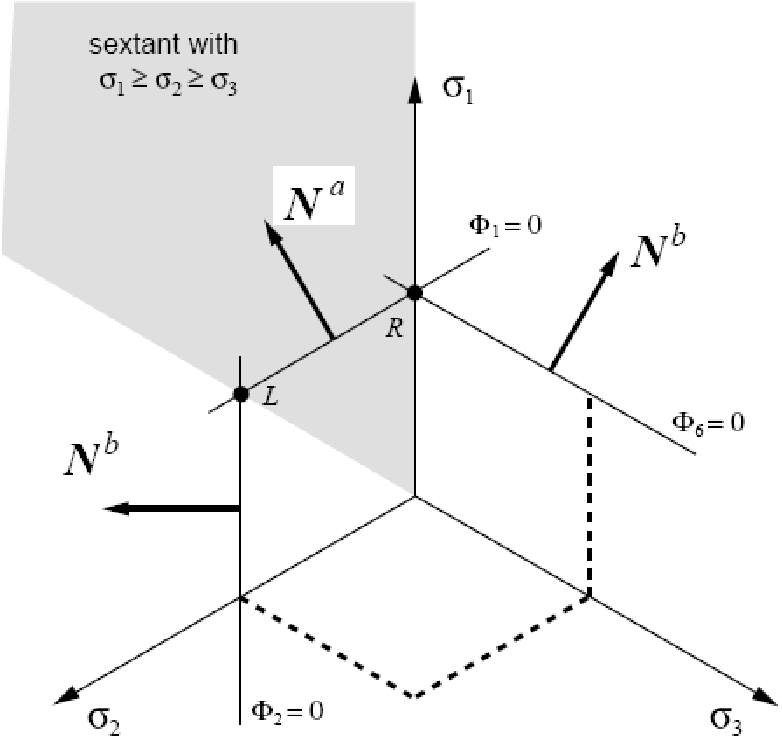
$$\Phi \leq 0, \quad \dot{\gamma} \geq 0, \quad \dot{\gamma}\Phi = 0$$

Multi-surface representation

$$\dot{\varepsilon}^p = \sum_{i=1}^{n} \dot{\gamma}_i \, \boldsymbol{N}_i$$

$$\Phi_i \leq 0, \quad \dot{\gamma}_i \geq 0, \quad \Phi_i \dot{\gamma}_i = 0$$

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$$



Possible representations:

1. Flow from **main plane**

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \, \boldsymbol{N}^a$$

$$\begin{aligned} \boldsymbol{N}^a &= \frac{\partial \Phi_1}{\partial \boldsymbol{\sigma}} = \frac{\partial}{\partial \boldsymbol{\sigma}}(\sigma_1 - \sigma_3) \\ &= \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \end{aligned}$$

2. Flow from **right corner**

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}^a \, \boldsymbol{N}^a + \dot{\gamma}^b \, \boldsymbol{N}^b$$

$$\boldsymbol{N}^b = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

3. Flow from **left corner**

$$\boldsymbol{N}^b = \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Associative Tresca isotropic hardening law

Yield stress

$$\sigma_y = \sigma_y(\bar{\varepsilon}^p)$$

General accumulated plastic strain evolution for multi-surface models

$$\dot{\bar{\varepsilon}}^p = -\sum_{i=1}^{n} \dot{\gamma}^i \frac{\partial \Phi_i}{\partial \kappa}$$

n = number of **active surfaces**

Tresca model

$$\dot{\bar{\varepsilon}}^p = -\dot{\gamma} \frac{\partial \Phi_1}{\partial \kappa} = \dot{\gamma}$$

main plane

$$\dot{\bar{\varepsilon}}^p = -\dot{\gamma}^a \frac{\partial \Phi_1}{\partial \kappa} - \dot{\gamma}^b \frac{\partial \Phi_6}{\partial \kappa} = \dot{\gamma}^a + \dot{\gamma}^b$$

right corner

$$\dot{\bar{\varepsilon}}^p = -\dot{\gamma}^a \frac{\partial \Phi_1}{\partial \kappa} - \dot{\gamma}^b \frac{\partial \Phi_2}{\partial \kappa} = \dot{\gamma}^a + \dot{\gamma}^b$$

left corner

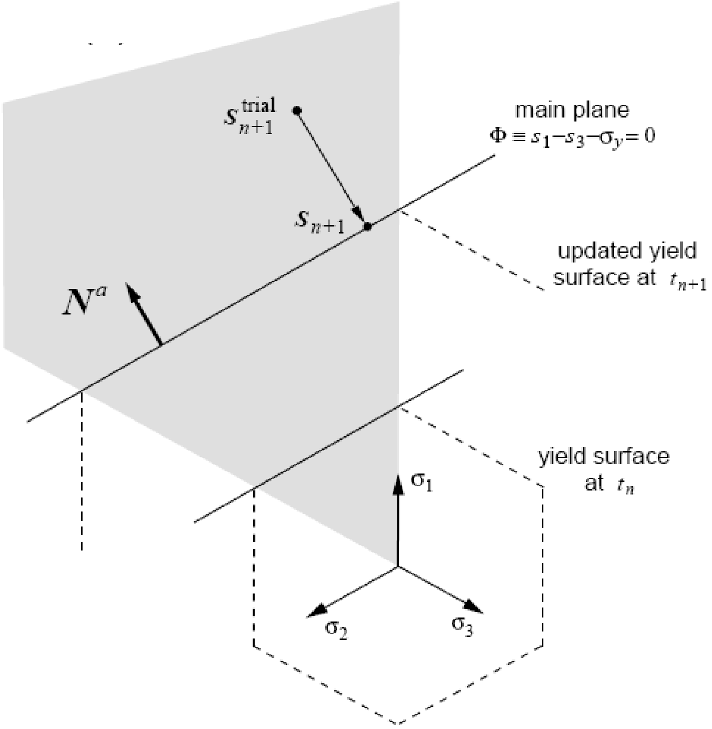
Three possible return mappings

1. Main plane (one equation)

$$\tilde{\Phi}(\Delta\gamma) \equiv s_1^{\mathrm{trial}} - s_3^{\mathrm{trial}} - 4G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}_n^p + \Delta\gamma) = 0$$

Principal stress update

$$s_1 = s_1^{\mathrm{trial}} - 2G\,\Delta\gamma$$
$$s_2 = s_2^{\mathrm{trial}}$$
$$s_3 = s_3^{\mathrm{trial}} + 2G\,\Delta\gamma$$



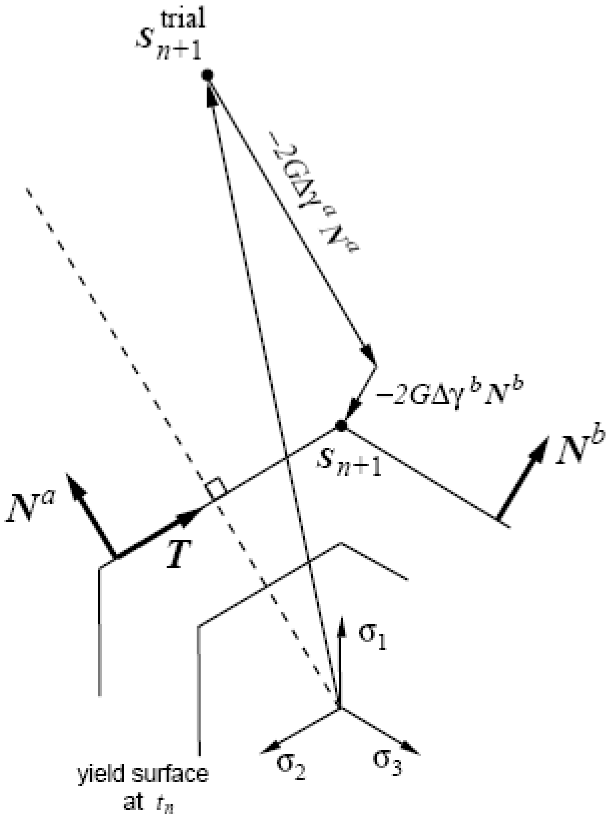
2. Right corner (**two** equations – ***2-vector return mapping***)

$$\tilde{\Phi}^a(\Delta\gamma^a,\Delta\gamma^b) \equiv s_1^{\text{trial}} - s_3^{\text{trial}} - 2G(2\Delta\gamma^a + \Delta\gamma^b) - \tilde{\sigma}_y(\Delta\gamma^a,\Delta\gamma^b) = 0$$

$$\tilde{\Phi}^b(\Delta\gamma^a,\Delta\gamma^b) \equiv s_1^{\text{trial}} - s_2^{\text{trial}} - 2G(\Delta\gamma^a + 2\Delta\gamma^b) - \tilde{\sigma}_y(\Delta\gamma^a,\Delta\gamma^b) = 0$$

Principal stress update

$$\begin{aligned} s_1 &= s_1^{\text{trial}} - 2G\,(\Delta\gamma^a + \Delta\gamma^b) \\ s_2 &= s_2^{\text{trial}} + 2G\,\Delta\gamma^b \\ s_3 &= s_3^{\text{trial}} + 2G\,\Delta\gamma^a\,. \end{aligned}$$



3. Left corner (**two** equations – ***2-vector return mapping***)

$$\tilde{\Phi}^a(\Delta\gamma^a, \Delta\gamma^b) \equiv s_1^{\text{trial}} - s_3^{\text{trial}} - 2G(2\Delta\gamma^a + \Delta\gamma^b) - \tilde{\sigma}_y(\Delta\gamma^a, \Delta\gamma^b) = 0$$

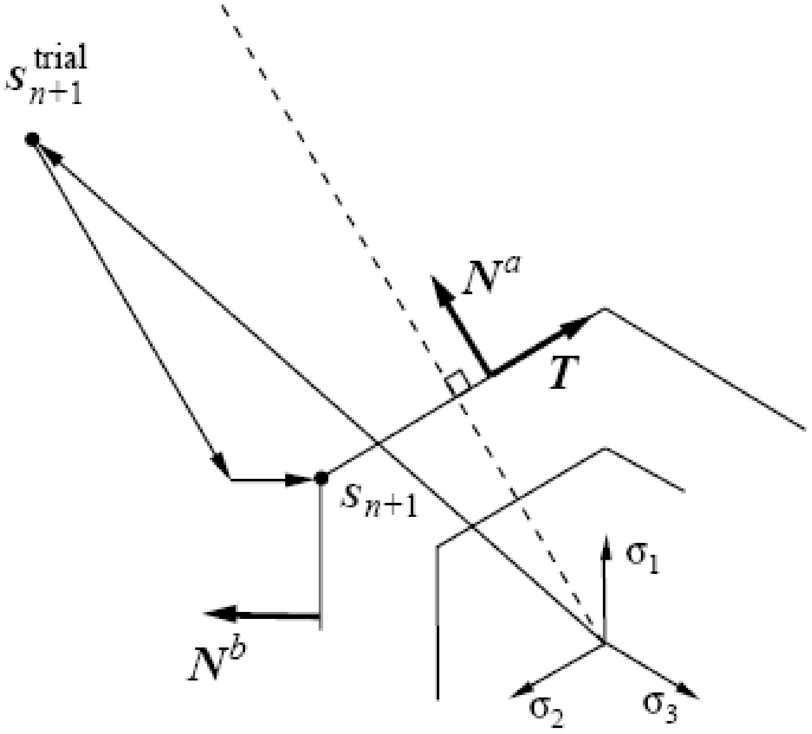
$$\tilde{\Phi}^b(\Delta\gamma^a, \Delta\gamma^b) \equiv s_2^{\text{trial}} - s_3^{\text{trial}} - 2G(\Delta\gamma^a + 2\Delta\gamma^b) - \tilde{\sigma}_y(\Delta\gamma^a, \Delta\gamma^b) = 0$$

Principal stress update

$$s_1 = s_1^{\text{trial}} - 2G\,\Delta\gamma^a$$

$$s_2 = s_2^{\text{trial}} - 2G\,\Delta\gamma^b$$

$$s_3 = s_3^{\text{trial}} + 2G\,(\Delta\gamma^a + \Delta\gamma^b)$$

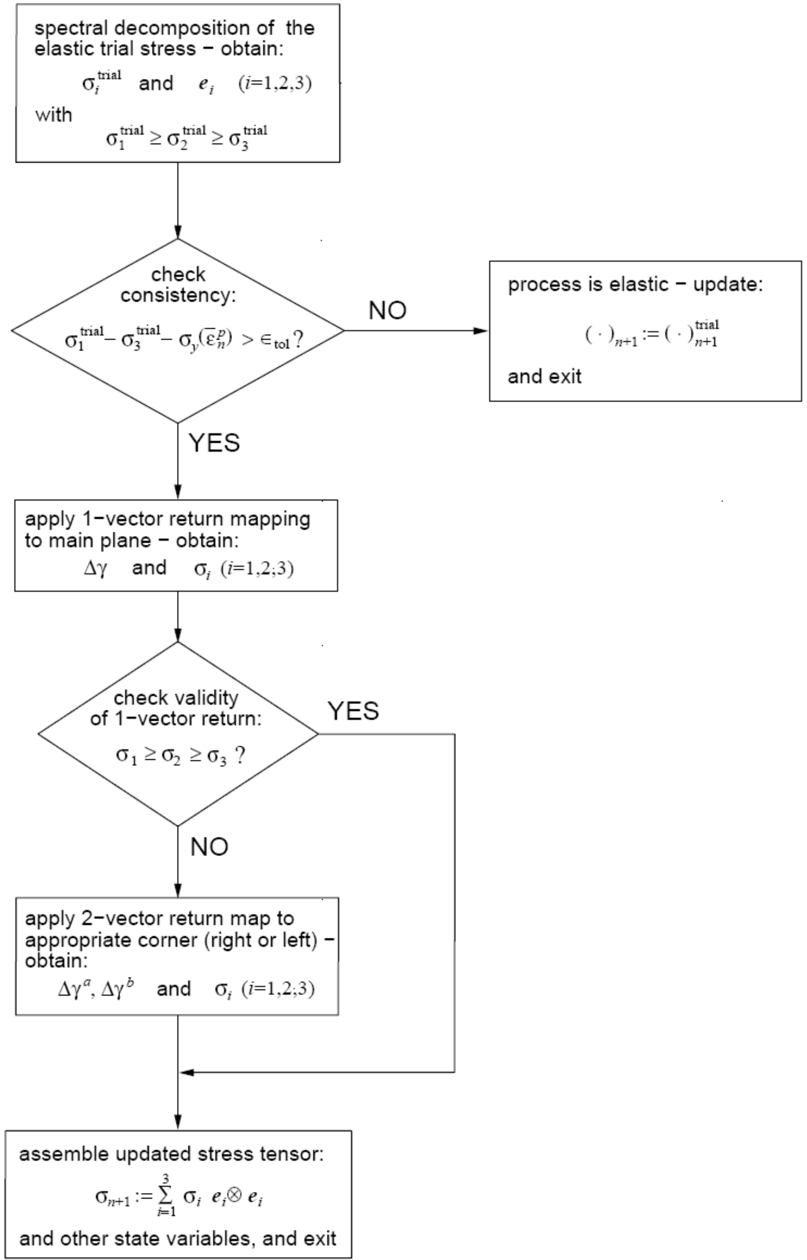


…Finally, the stress **tensor** update reads

$$\boldsymbol{\sigma}_{n+1} := \sum_{i=1}^{3} (s_i + p_{n+1})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

FLOWCHART

Tresca in principal stresses



spectral decomposition of the elastic trial stress – obtain:

σ_i^{trial} and e_i $(i=1,2,3)$

with

$\sigma_1^{\text{trial}} \geq \sigma_2^{\text{trial}} \geq \sigma_3^{\text{trial}}$

check consistency:

$\sigma_1^{\text{trial}} - \sigma_3^{\text{trial}} - \sigma_y(\bar{\varepsilon}_n^p) > \epsilon_{\text{tol}}$?

NO

process is elastic – update:

$(\cdot)_{n+1} := (\cdot)_{n+1}^{\text{trial}}$

and exit

YES

apply 1–vector return mapping to main plane – obtain:

$\Delta\gamma$ and σ_i $(i=1,2,3)$

check validity of 1–vector return:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$?

YES

NO

apply 2–vector return map to appropriate corner (right or left) – obtain:

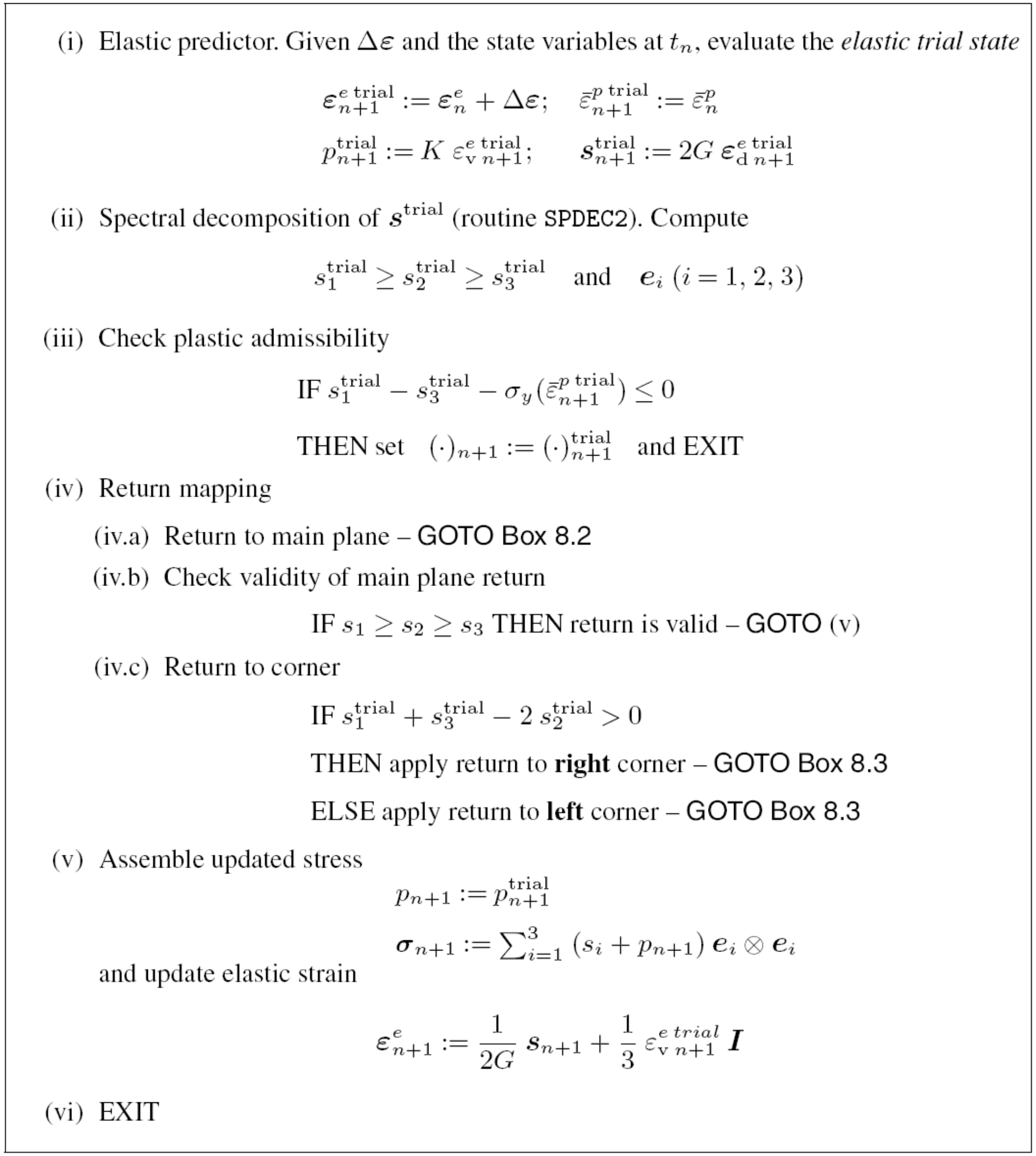
$\Delta\gamma^a, \Delta\gamma^b$ and σ_i $(i=1,2,3)$

assemble updated stress tensor:

$$\sigma_{n+1} := \sum_{i=1}^{3} \sigma_i \; e_i \otimes e_i$$

and other state variables, and exit

Computational Plasticity



(i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n, evaluate the *elastic trial state*

$$\boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1} := \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}; \qquad \bar{\varepsilon}^{p\,\mathrm{trial}}_{n+1} := \bar{\varepsilon}^{p}_{n}$$

$$p^{\mathrm{trial}}_{n+1} := K\,\varepsilon^{e\,\mathrm{trial}}_{\mathrm{v}\;n+1}; \qquad \boldsymbol{s}^{\mathrm{trial}}_{n+1} := 2G\,\boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{\mathrm{d}\;n+1}$$

(ii) Spectral decomposition of $\boldsymbol{s}^{\mathrm{trial}}$ (routine `SPDEC2`). Compute

$$s^{\mathrm{trial}}_1 \geq s^{\mathrm{trial}}_2 \geq s^{\mathrm{trial}}_3 \quad \text{and} \quad \boldsymbol{e}_i \; (i = 1, 2, 3)$$

(iii) Check plastic admissibility

IF $s^{\mathrm{trial}}_1 - s^{\mathrm{trial}}_3 - \sigma_y(\bar{\varepsilon}^{p\,\mathrm{trial}}_{n+1}) \leq 0$

THEN set $(\cdot)_{n+1} := (\cdot)^{\mathrm{trial}}_{n+1}$ and EXIT

(iv) Return mapping

(iv.a) Return to main plane – GOTO Box 8.2

(iv.b) Check validity of main plane return

IF $s_1 \geq s_2 \geq s_3$ THEN return is valid – GOTO (v)

(iv.c) Return to corner

IF $s^{\mathrm{trial}}_1 + s^{\mathrm{trial}}_3 - 2\,s^{\mathrm{trial}}_2 > 0$

THEN apply return to **right** corner – GOTO Box 8.3

ELSE apply return to **left** corner – GOTO Box 8.3

(v) Assemble updated stress

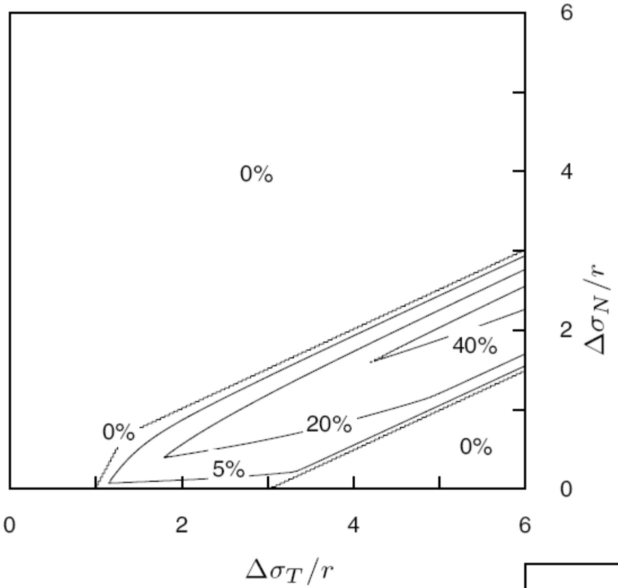
$$p_{n+1} := p^{\mathrm{trial}}_{n+1}$$

$$\boldsymbol{\sigma}_{n+1} := \textstyle\sum_{i=1}^{3} (s_i + p_{n+1})\,\boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

and update elastic strain

$$\boldsymbol{\varepsilon}^{e}_{n+1} := \frac{1}{2G}\,\boldsymbol{s}_{n+1} + \frac{1}{3}\,\varepsilon^{e\,trial}_{\mathrm{v}\;n+1}\,\boldsymbol{I}$$

(vi) EXIT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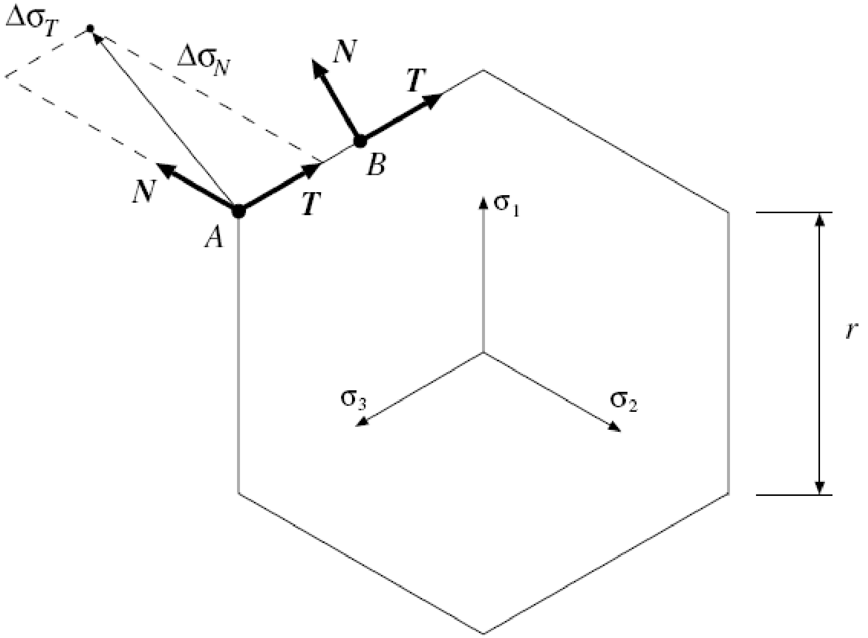
(a)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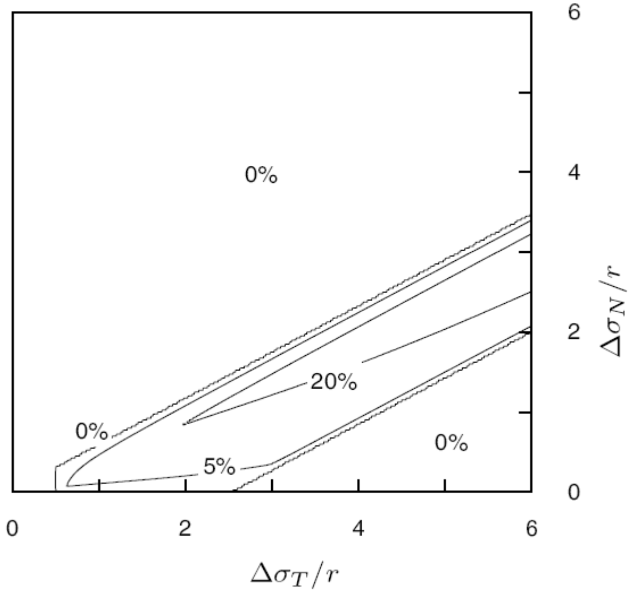
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(b)

Consistent elasto-plastic tangent operator

$$\mathsf{D}^{ep} \equiv \frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}}$$ under incremental plastic straining

The (principal stress) return mapping-based algorithm defines the updated stress tensor as an ***isotropic tensor function of the elastic trial strain tensor.*** The eigenvalues of the stress tensor are implicit functions of the eigenvalues of the trial strain defined by the return algorithm:

$$\sigma_i = \tilde{\sigma}_i(\varepsilon_1^{e\ \mathrm{trial}}, \varepsilon_2^{e\ \mathrm{trial}}, \varepsilon_3^{e\ \mathrm{trial}}), \qquad i = 1, 2, 3$$

We compute the partial derivatives (consistently with the relevant return mapping applied):

$$\partial\sigma_i / \partial\varepsilon_j^{e\ \mathrm{trial}}$$

and then assemble D^{ep} according to a general formula for isotropic tensor function derivatives.

$$\frac{\partial \sigma_i}{\partial \varepsilon_j^{e\ \mathrm{trial}}} = \frac{\partial s_i}{\partial \varepsilon_{\mathrm{d}\ k}^{e\ \mathrm{trial}}}(\delta_{kj} - \tfrac{1}{3}) + K$$

…for example, for the right corner return…

$$\left[\frac{\partial s_i}{\partial \varepsilon_{\mathrm{d}\ j}^{e\ \mathrm{trial}}}\right] = \begin{bmatrix} 2G\left(1 - \dfrac{8G^2}{\det \mathbf{d}}\right) & \dfrac{4G^2}{\det \mathbf{d}}(d_{ab} - d_{aa}) & \dfrac{4G^2}{\det \mathbf{d}}(d_{ba} - d_{bb}) \\ \dfrac{8G^3}{\det \mathbf{d}} & 2G\left(1 + \dfrac{2G\, d_{aa}}{\det \mathbf{d}}\right) & -\dfrac{4G^2}{\det \mathbf{d}} d_{ba} \\ \dfrac{8G^3}{\det \mathbf{d}} & -\dfrac{4G^2}{\det \mathbf{d}} d_{ab} & 2G\left(1 + \dfrac{2G\, d_{bb}}{\det \mathbf{d}}\right) \end{bmatrix}$$

$$d_{aa} = -4G - H \qquad d_{ab} = -2G - H$$

$$d_{ba} = -2G - H \qquad d_{bb} = -4G - H$$

Return-mapping-based Integration algorithm for the Mohr-Coulomb Model

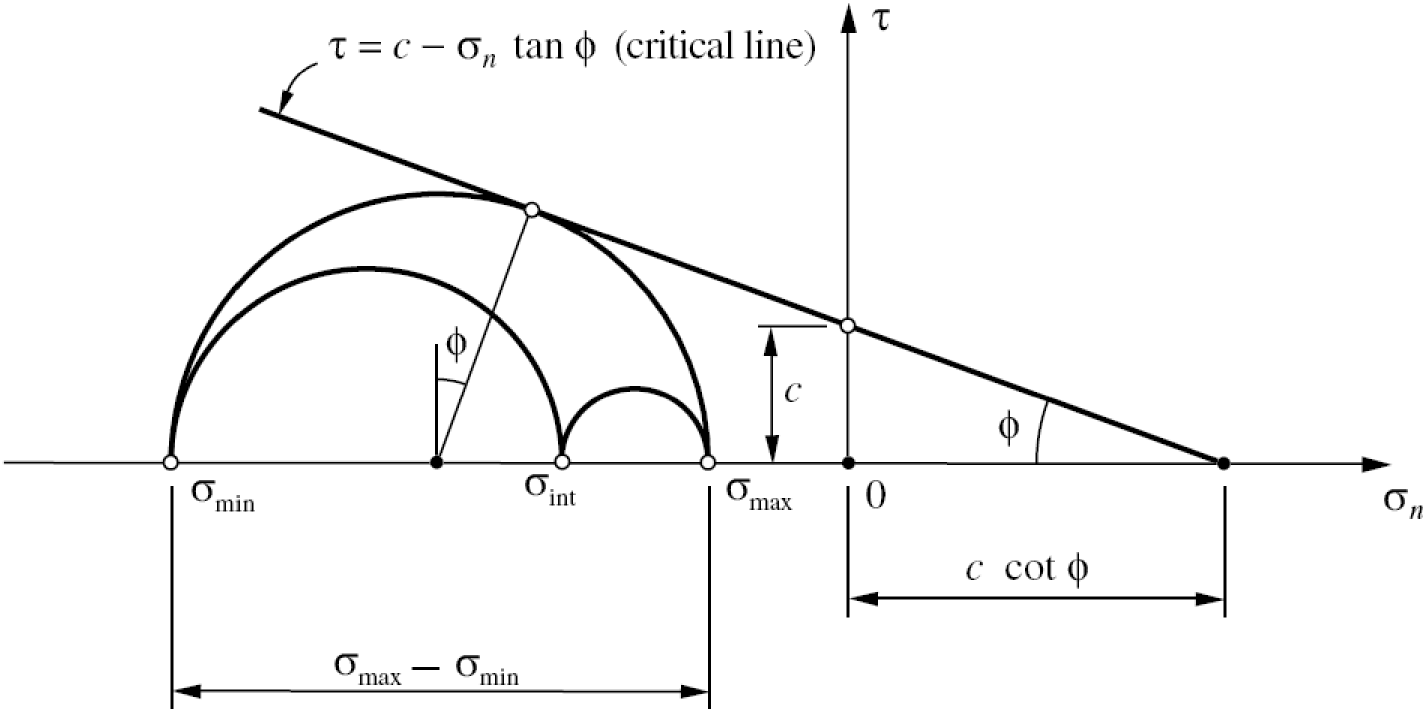
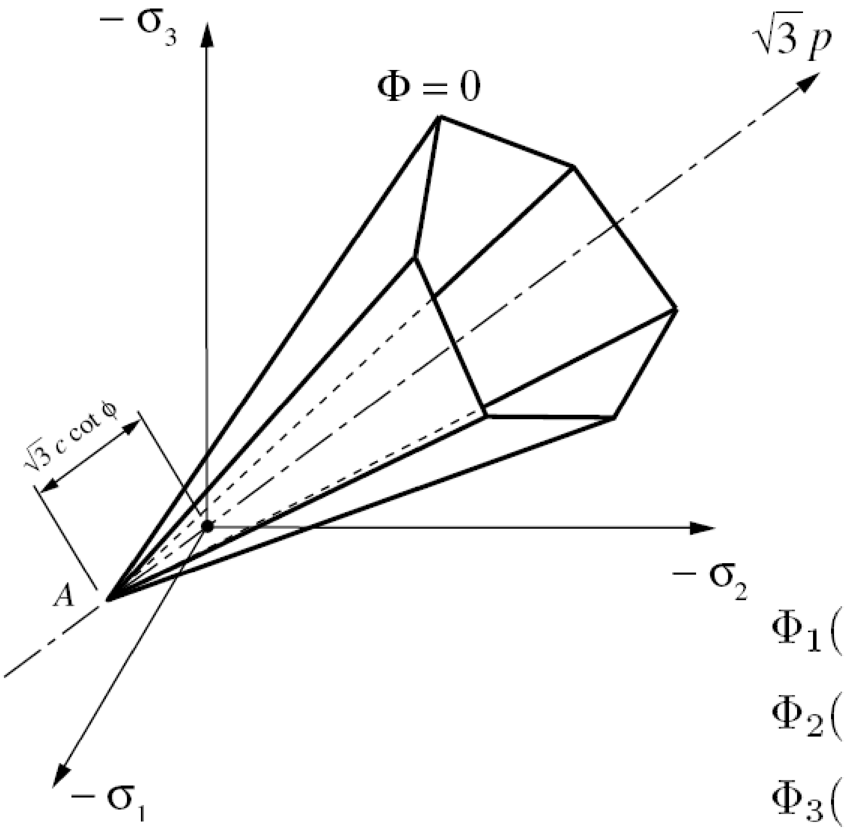


Figure 6.12. The Mohr–Coulomb criterion. Mohr plane representation.

Multi-surface representation



$$\Phi_1(\boldsymbol{\sigma}, c) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi - 2\,c\,\cos\phi$$

$$\Phi_2(\boldsymbol{\sigma}, c) = \sigma_2 - \sigma_3 + (\sigma_2 + \sigma_3)\sin\phi - 2\,c\,\cos\phi$$

$$\Phi_3(\boldsymbol{\sigma}, c) = \sigma_2 - \sigma_1 + (\sigma_2 + \sigma_1)\sin\phi - 2\,c\,\cos\phi$$

$$\Phi_4(\boldsymbol{\sigma}, c) = \sigma_3 - \sigma_1 + (\sigma_3 + \sigma_1)\sin\phi - 2\,c\,\cos\phi$$

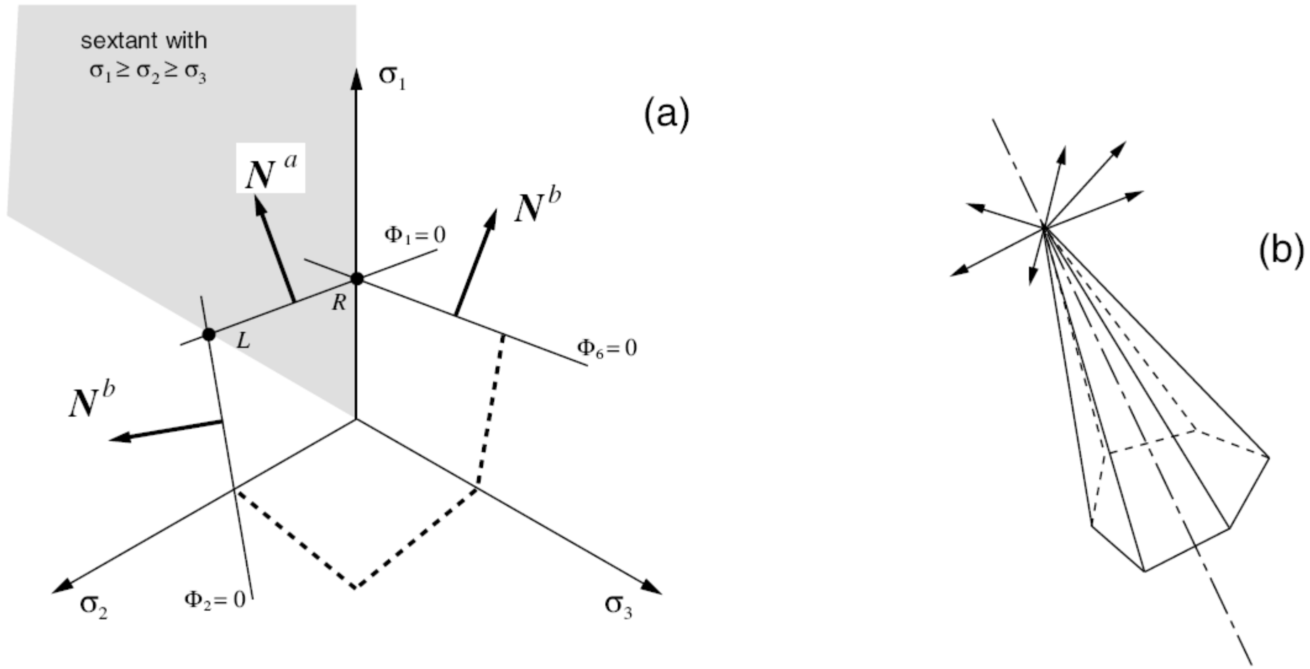
$$\Phi_5(\boldsymbol{\sigma}, c) = \sigma_3 - \sigma_2 + (\sigma_3 + \sigma_2)\sin\phi - 2\,c\,\cos\phi$$

$$\Phi_6(\boldsymbol{\sigma}, c) = \sigma_1 - \sigma_2 + (\sigma_1 + \sigma_2)\sin\phi - 2\,c\,\cos\phi$$

Associative/non-associative flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \sum_{i=1}^{6} \dot{\gamma}^i \boldsymbol{N}^i = \sum_{i=1}^{6} \dot{\gamma}^i \frac{\partial \Phi_i}{\partial \boldsymbol{\sigma}}$$

$$\dot{\varepsilon}^p_v \equiv \dot{\varepsilon}^p_1 + \dot{\varepsilon}^p_2 + \dot{\varepsilon}^p_3 = 2\ \sin\phi \sum_{i=1}^{6} \dot{\gamma}^i$$



Main plane

$$\dot{\varepsilon}^p = \dot{\gamma}\, \boldsymbol{N}^a \qquad \boldsymbol{N}^a = \frac{\partial \Phi_1}{\partial \boldsymbol{\sigma}} = \frac{\partial}{\partial \boldsymbol{\sigma}}\left[\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi\right]$$

$$= (1+\sin\phi)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - (1-\sin\phi)\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Corners

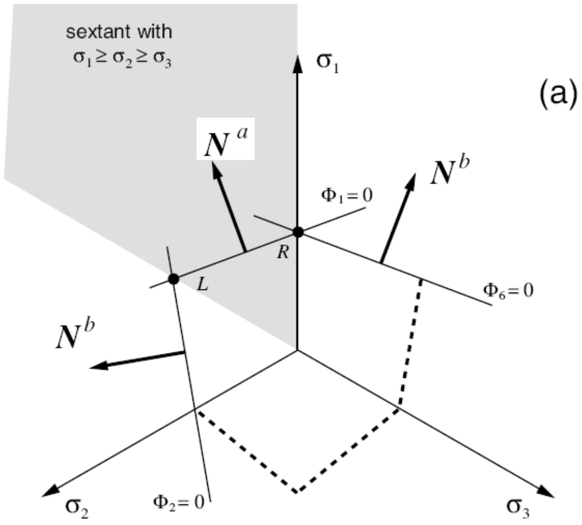
$$\dot{\varepsilon}^p = \dot{\gamma}^a\, \boldsymbol{N}^a + \dot{\gamma}^b\, \boldsymbol{N}^b$$

…right

$$\boldsymbol{N}^b = (1+\sin\phi)\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - (1-\sin\phi)\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

…left

$$\boldsymbol{N}^b = (1+\sin\phi)\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - (1-\sin\phi)\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$



Apex

$$\dot{\varepsilon}^p = \sum_{i=1}^{6} \dot{\gamma}^i\, \boldsymbol{N}^i$$

Associative hardening

$$\dot{\bar{\varepsilon}}^p = 2\cos\phi \sum_{i=1}^{6} \dot{\gamma}^i$$

Main plane

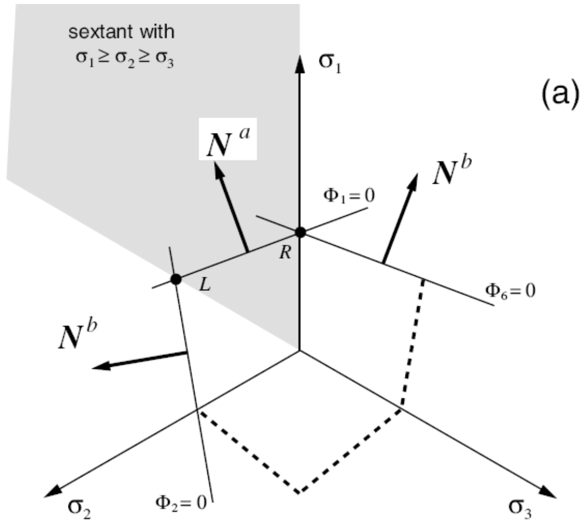
$$\dot{\bar{\varepsilon}}^p = 2\cos\phi\,\dot{\gamma}$$

Corners

$$\dot{\bar{\varepsilon}}^p = 2\cos\phi\,(\dot{\gamma}^a + \dot{\gamma}^b)$$

Apex

$$\dot{\bar{\varepsilon}}^p = \frac{\cos\phi}{\sin\psi}\,\dot{\varepsilon}^p_{\mathrm{v}}$$



Elastic predictor/return mapping algorithm

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} - \mathsf{D}^{e} : \Delta\boldsymbol{\varepsilon}^{p}$$

Multi-surface representation

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} - \mathsf{D}^{e} : \sum_{i=1}^{6} \Delta\gamma^{i}\,\boldsymbol{N}_{n+1}^{i}$$

In principal stresses

$$\sigma_j = \sigma_j^{\mathrm{trial}} - \sum_{i=1}^{6} \Delta\gamma^{i}\,(2G\,[\boldsymbol{N}_{\mathrm{d}}^{i}]_j - K\,N_{\mathrm{v}}^{i})$$

Return mapping to main plane

$$\sigma_1 = \sigma_1^{\text{trial}} - \Delta\gamma\,[2G(1+\tfrac{1}{3}\sin\psi) + 2K\ \sin\psi]$$

$$\sigma_2 = \sigma_2^{\text{trial}} + \Delta\gamma\,(\tfrac{4}{3}G - 2K)\sin\psi$$

$$\sigma_3 = \sigma_3^{\text{trial}} + \Delta\gamma\,[2G(1-\tfrac{1}{3}\sin\psi) - 2K\ \sin\psi]$$

$$\Delta\bar{\varepsilon}^p = 2\cos\phi\,\Delta\gamma$$

return mapping equation

$$\begin{aligned}\tilde{\Phi}(\Delta\gamma) \equiv (\sigma_1^{\text{trial}} - \sigma_3^{\text{trial}}) + (\sigma_1^{\text{trial}} + \sigma_3^{\text{trial}})\sin\phi \\ - 2\,c(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}^p)\cos\phi - a\Delta\gamma = 0,\end{aligned}$$

$$a = 4G(1+\tfrac{1}{3}\sin\phi\sin\psi) + 4K\sin\phi\sin\psi$$

Return mapping to right edge

$$\Delta \boldsymbol{\varepsilon}^p = \Delta\gamma^a \boldsymbol{N}^a + \Delta\gamma^b \boldsymbol{N}^b$$

$$\sigma_1 = \sigma_1^{\text{trial}} - [2G(1 + \tfrac{1}{3}\sin\psi) + 2K\ \sin\psi](\Delta\gamma^a + \Delta\gamma^b)$$

$$\sigma_2 = \sigma_2^{\text{trial}} + (\tfrac{4}{3}G - 2K)\sin\psi\ \Delta\gamma^a + [2G(1 - \tfrac{1}{3}\sin\psi) - 2K\ \sin\psi]\Delta\gamma^b$$

$$\sigma_3 = \sigma_3^{\text{trial}} + [2G(1 - \tfrac{1}{3}\sin\psi) - 2K\ \sin\psi]\Delta\gamma^a + (\tfrac{4}{3}G - 2K)\sin\psi\ \Delta\gamma^b$$

$$\Delta\bar{\varepsilon}^p = 2\cos\phi\,(\Delta\gamma^a + \Delta\gamma^b)$$

return mapping equations

$$\begin{aligned}\tilde{\Phi}^a(\Delta\gamma^a, \Delta\gamma^b) \equiv{}& \sigma_1^{\text{trial}} - \sigma_3^{\text{trial}} + (\sigma_1^{\text{trial}} + \sigma_3^{\text{trial}})\sin\phi \\ & - 2\cos\phi\, c(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}^p) - a\,\Delta\gamma^a - b\,\Delta\gamma^b = 0\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^b(\Delta\gamma^a, \Delta\gamma^b) \equiv{}& \sigma_2^{\text{trial}} - \sigma_3^{\text{trial}} + (\sigma_2^{\text{trial}} + \sigma_3^{\text{trial}})\sin\phi \\ & - 2\cos\phi\, c(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}^p) - b\,\Delta\gamma^a - a\,\Delta\gamma^b = 0\end{aligned}$$

$$b = 2G(1 + \sin\phi + \sin\psi - \tfrac{1}{3}\sin\phi\sin\psi) + 4K\sin\phi\sin\psi$$

Return mapping to left edge

$$\sigma_1 = \sigma_1^{\text{trial}} - [2G(1 + \tfrac{1}{3}\sin\psi) + 2K\ \sin\psi]\Delta\gamma^a + (\tfrac{4}{3}G - 2K)\sin\psi\ \Delta\gamma^b$$

$$\sigma_2 = \sigma_2^{\text{trial}} + (\tfrac{4}{3}G - 2K)\sin\psi\ \Delta\gamma^a - [2G(1 + \tfrac{1}{3}\sin\psi) + 2K\ \sin\psi]\Delta\gamma^b$$

$$\sigma_3 = \sigma_3^{\text{trial}} + [2G(1 - \tfrac{1}{3}\sin\psi) - 2K\ \sin\psi](\Delta\gamma^a + \Delta\gamma^b),$$

$$\Delta\bar{\varepsilon}^p = 2\cos\phi\,(\Delta\gamma^a + \Delta\gamma^b)$$

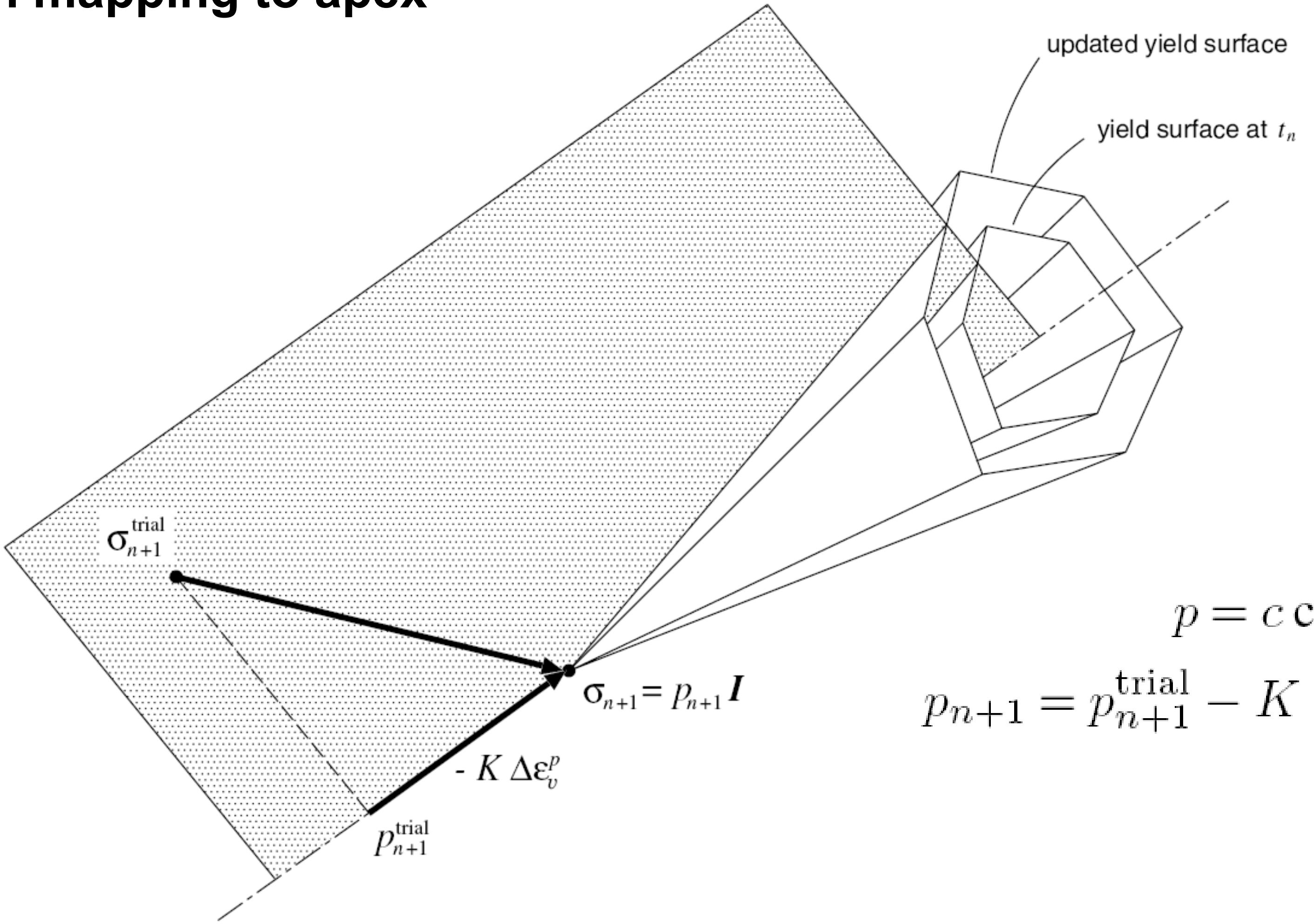
return mapping equations

$$\begin{aligned}\tilde{\Phi}^a(\Delta\gamma^a, \Delta\gamma^b) \equiv \sigma_1^{\text{trial}} - \sigma_3^{\text{trial}} + (\sigma_1^{\text{trial}} + \sigma_3^{\text{trial}})\sin\phi \\ - 2\cos\phi\, c(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}^p) - a\,\Delta\gamma^a - b\,\Delta\gamma^b = 0\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^b(\Delta\gamma^a, \Delta\gamma^b) \equiv \sigma_2^{\text{trial}} - \sigma_3^{\text{trial}} + (\sigma_2^{\text{trial}} + \sigma_3^{\text{trial}})\sin\phi \\ - 2\cos\phi\, c(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}^p) - b\,\Delta\gamma^a - a\,\Delta\gamma^b = 0\end{aligned}$$

$$b = 2G(1 - \sin\phi - \sin\psi - \tfrac{1}{3}\sin\phi\sin\psi) + 4K\sin\phi\sin\psi$$

Return mapping to apex



$$p = c \cot \phi$$

$$p_{n+1} = p_{n+1}^{\text{trial}} - K\,\Delta\varepsilon_{\text{v}}^{p}$$

Figure 8.8. Mohr–Coulomb model. Return mapping to apex.

Return mapping to apex

$$p = c \cot \phi$$

$$p_{n+1} = p_{n+1}^{\text{trial}} - K \, \Delta \varepsilon_{\mathrm{v}}^{p}$$

$$c(\bar{\varepsilon}_n^p + \Delta \bar{\varepsilon}^p) \cot \phi - p_{n+1}^{\text{trial}} + K \, \Delta \varepsilon_{\mathrm{v}}^{p} = 0$$

$$\Delta \bar{\varepsilon}^p = \alpha \Delta \varepsilon_{\mathrm{v}}^{p}; \qquad \alpha \equiv \frac{\cos \phi}{\sin \psi}$$

return mapping equation

$$c(\bar{\varepsilon}_n^p + \alpha \Delta \varepsilon_{\mathrm{v}}^{p}) \cot \phi - p_{n+1}^{\text{trial}} + K \, \Delta \varepsilon_{\mathrm{v}}^{p} = 0$$

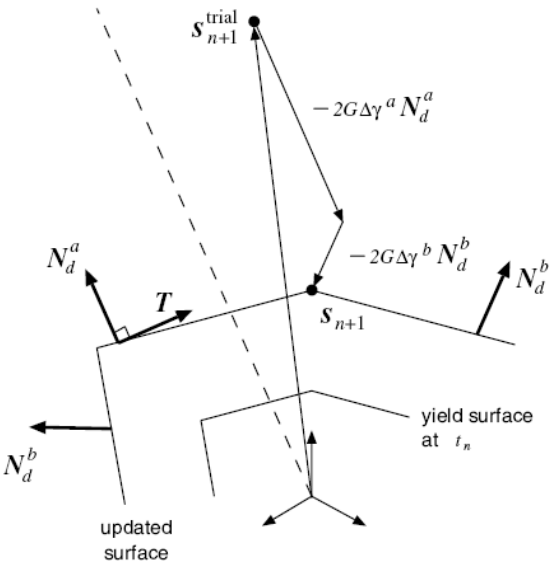
update of state

$$\bar{\varepsilon}_{n+1}^p := \bar{\varepsilon}_n^p + \alpha \Delta \varepsilon_v^p$$

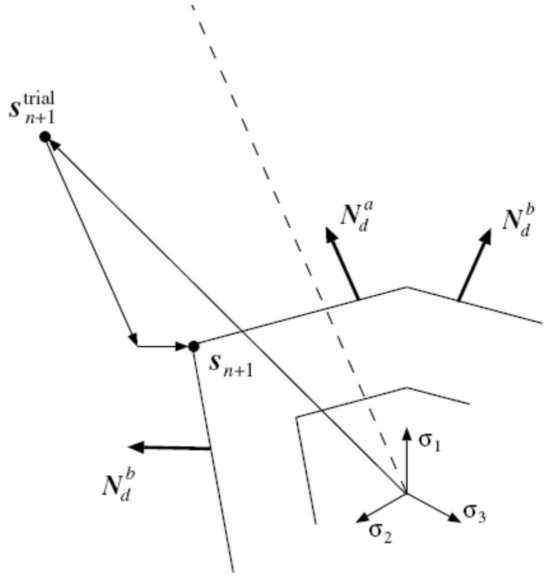
$$\boldsymbol{\sigma}_{n+1} := p_{n+1} \, \boldsymbol{I},$$

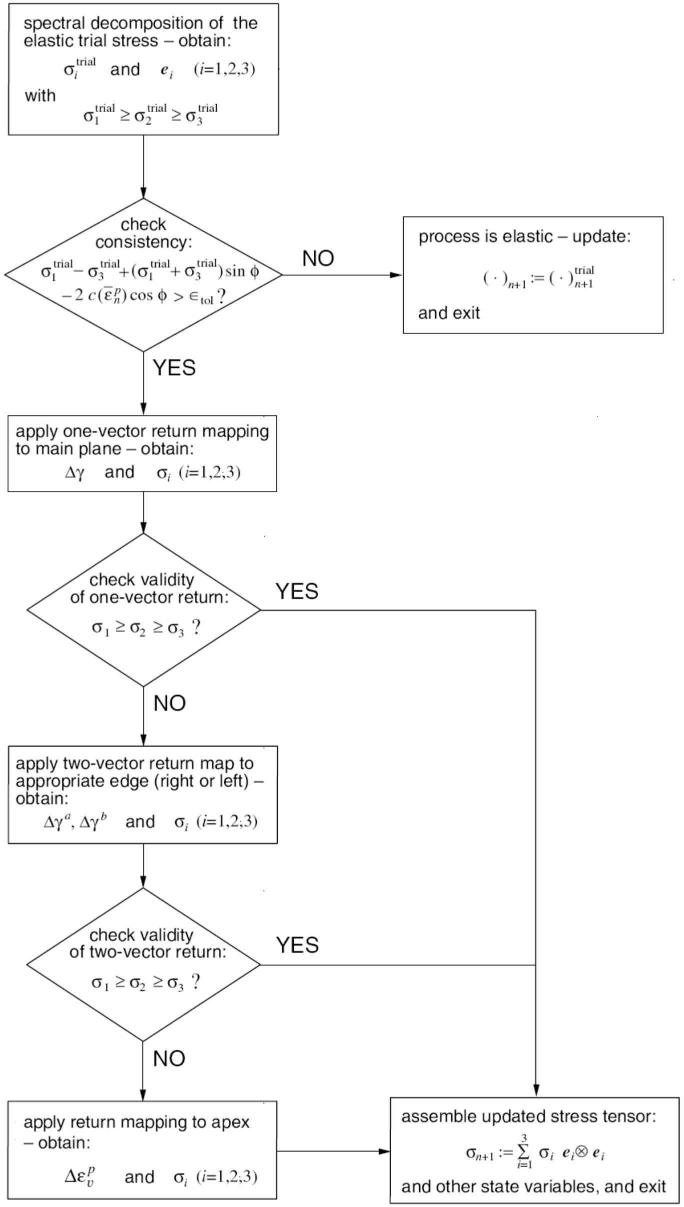
Selection of the correct return mapping

$\sigma_{n+1}^{\text{trial}} : \boldsymbol{T} > 0 \implies$ right



$\sigma_{n+1}^{\text{trial}} : \boldsymbol{T} < 0 \implies$ left





spectral decomposition of the elastic trial stress – obtain:

σ_i^{trial} and $\boldsymbol{e}_i$ $(i=1,2,3)$

with

$\sigma_1^{\text{trial}} \geq \sigma_2^{\text{trial}} \geq \sigma_3^{\text{trial}}$

check consistency:

$\sigma_1^{\text{trial}} - \sigma_3^{\text{trial}} + (\sigma_1^{\text{trial}} + \sigma_3^{\text{trial}}) \sin\phi - 2\, c(\bar{\varepsilon}_n^p) \cos\phi > \epsilon_{\text{tol}}\,?$

NO

process is elastic – update:

$(\,\cdot\,)_{n+1} := (\,\cdot\,)_{n+1}^{\text{trial}}$

and exit

YES

apply one-vector return mapping to main plane – obtain:

$\Delta\gamma$ and σ_i $(i=1,2,3)$

check validity of one-vector return:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$?

YES

NO

apply two-vector return map to appropriate edge (right or left) – obtain:

$\Delta\gamma^a, \Delta\gamma^b$ and σ_i $(i=1,2,3)$

check validity of two-vector return:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$?

YES

NO

apply return mapping to apex – obtain:

$\Delta\varepsilon_v^p$ and σ_i $(i=1,2,3)$

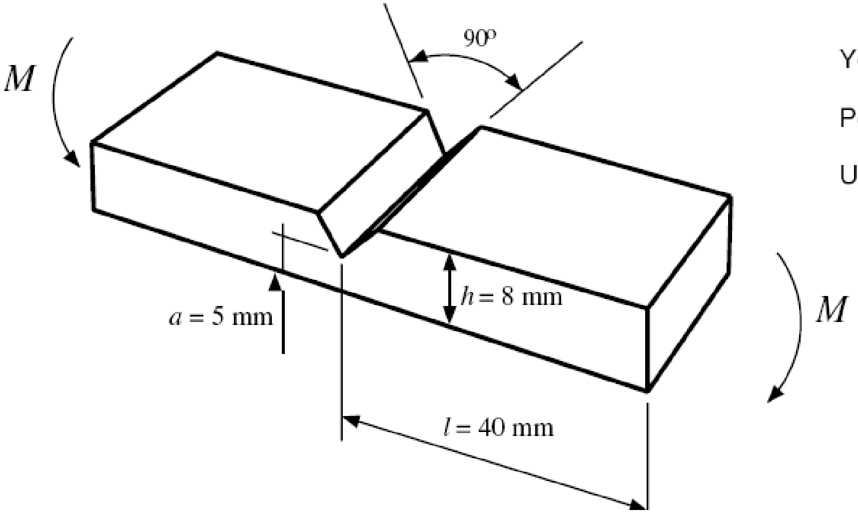
assemble updated stress tensor:

$\boldsymbol{\sigma}_{n+1} := \sum_{i=1}^{3} \sigma_i\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$

and other state variables, and exit

Consistent tangent

Analogous to Tresca in terms of principal stress derivatives

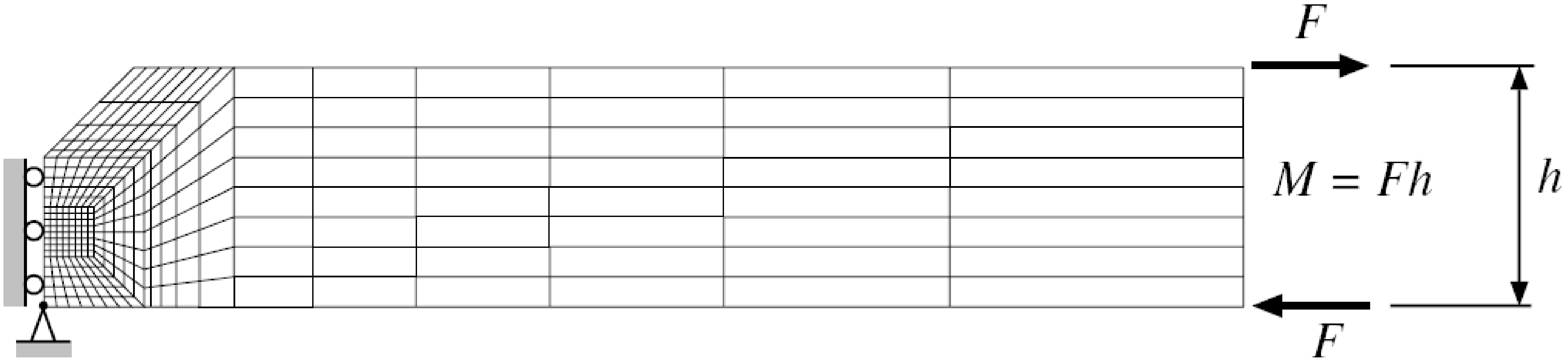


Material properties – Tresca model

Young's modulus: $E = 210$ GPa

Poisson ratio: $\nu = 0.3$

Uniaxial yield stress: $\sigma_y = 0.24$ GPa (perfectly plastic)



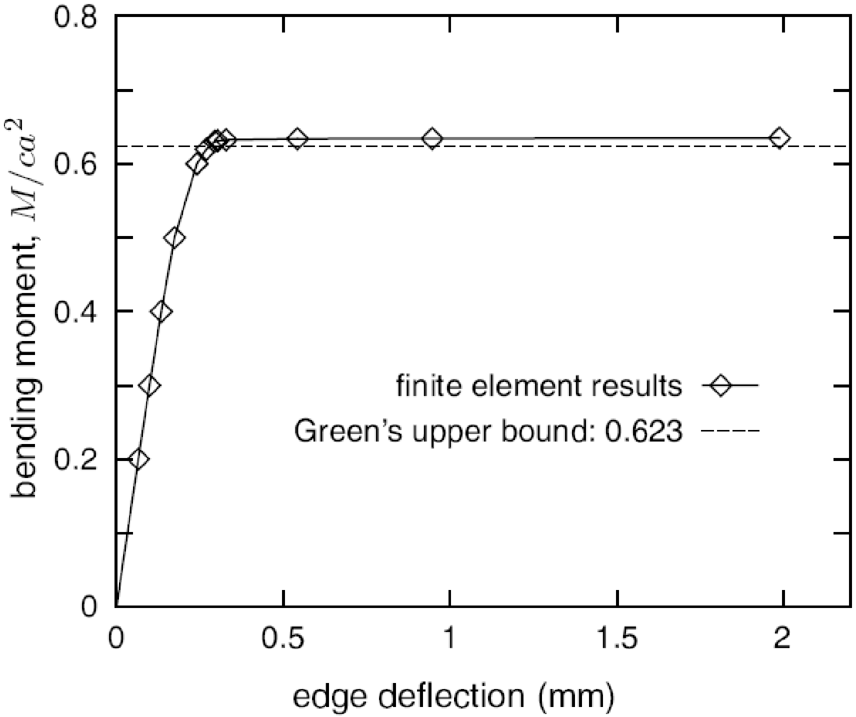
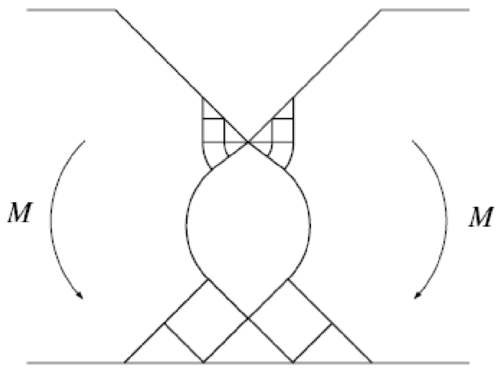
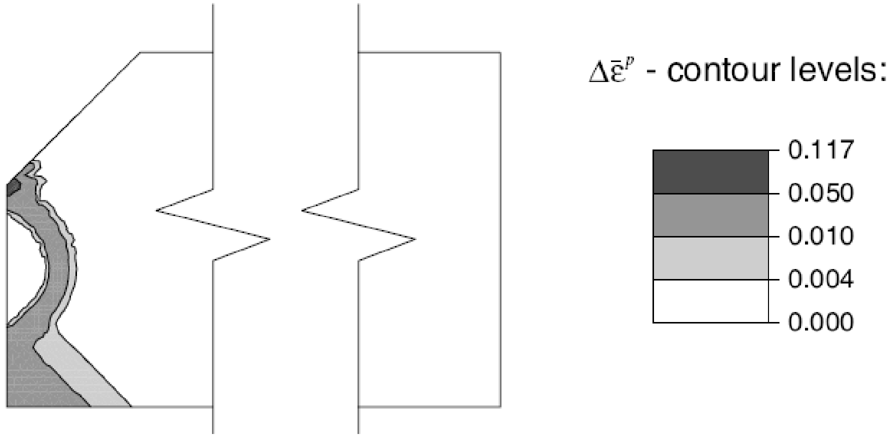
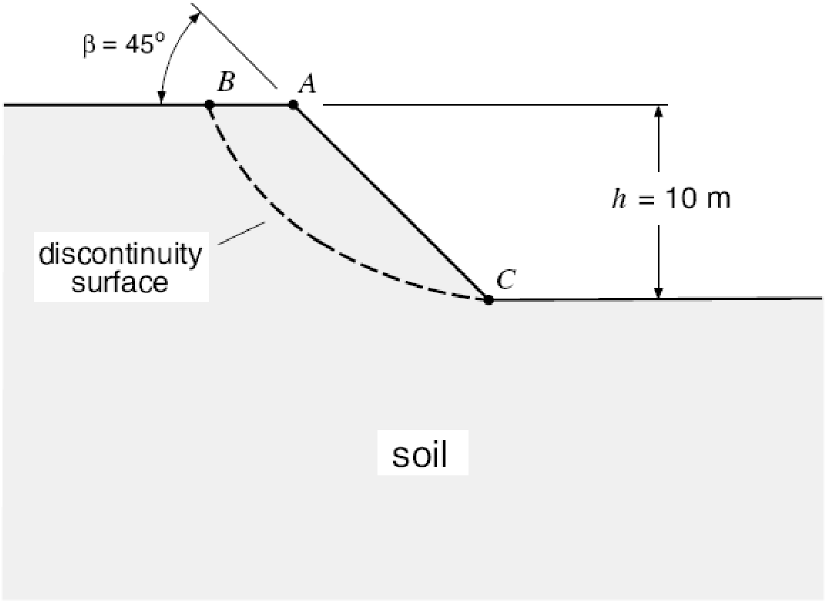


Figure 8.19. Bending of a V-notched bar. Moment-deflection diagram. (Reproduced with permissio from A new computational model for Tresca plasticity at finite strains with an optimal parametrizatio in the principal space, D Perić and EA de Souza Neto, *Computer Methods in Applied Mechanics an Engineering*, Vol 171 © 1999 Elsevier Science S.A.)


M
M


$\Delta\bar{\varepsilon}^p$ - contour levels:
0.117
0.050
0.010
0.004
0.000



Soil properties. Mohr–Coulomb model

Young's modulus: E = 20 000 kPa

Poisson's ratio: $\nu = 0.49$

Cohesion: c = 50 kPa

Internal friction angle: $\phi = 20^{o}$

Dilatancy angle: $\psi = 20^{o}$ (associative flow)

Specific weight: γ = 20 kN/m^3

Figure 8.30. Slope stability. Geometry and material properties.

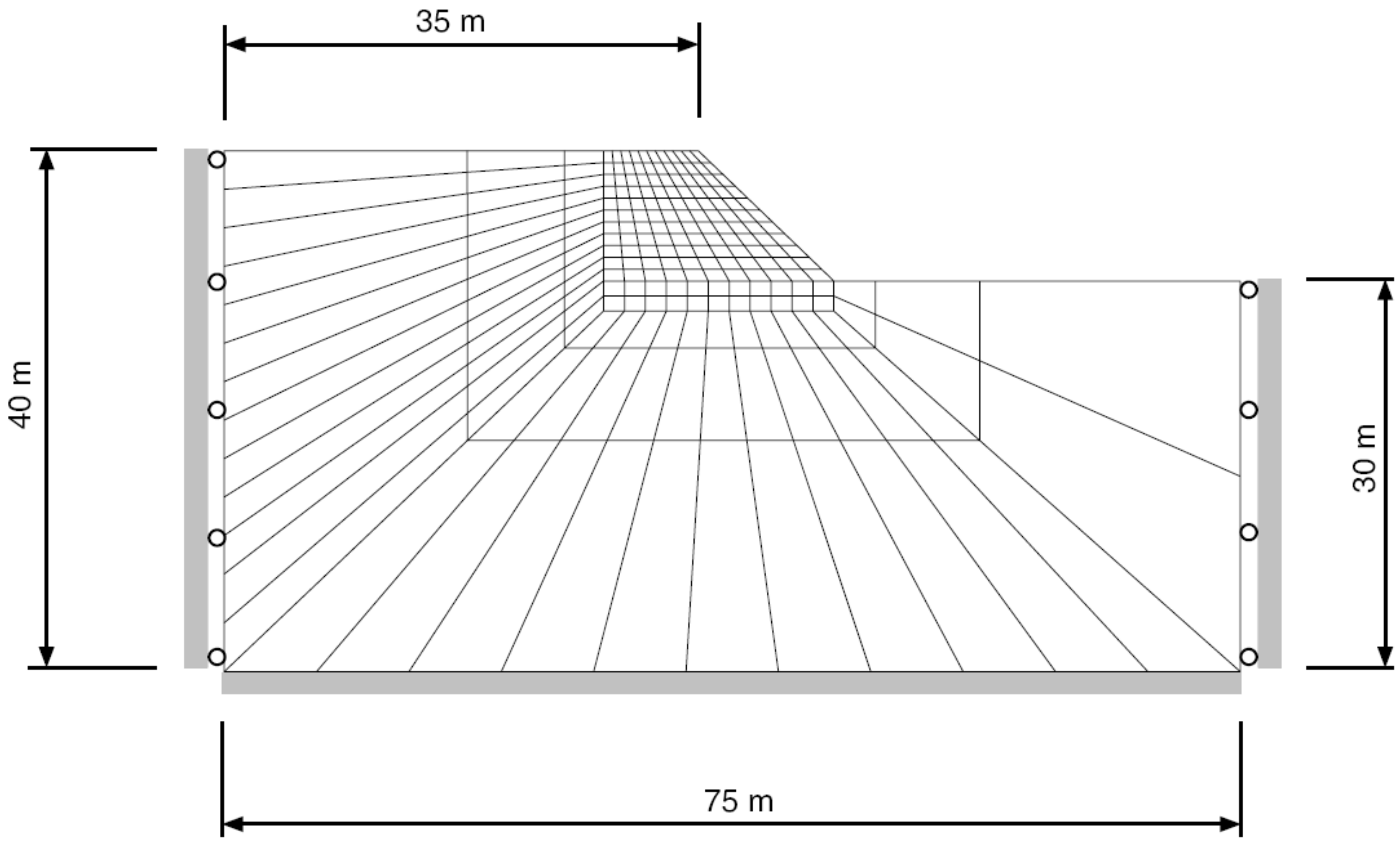
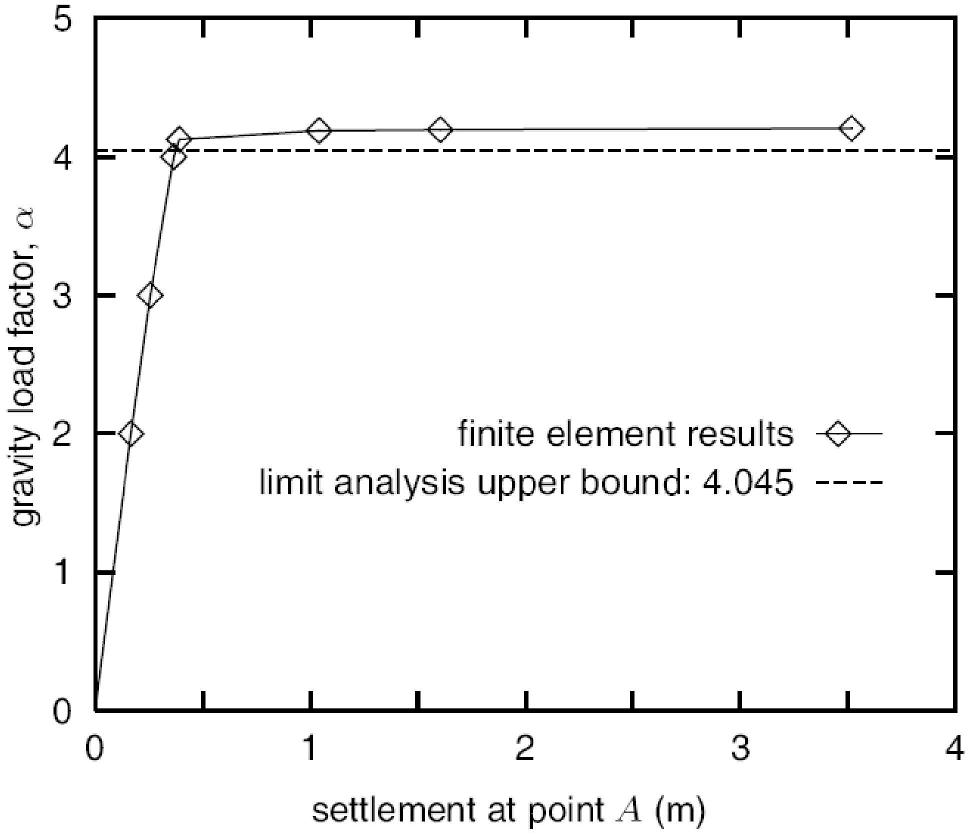


Figure 8.31. Slope stability. Finite element mesh.


gravity load factor, α
settlement at point A (m)
0
1
2
3
4
5
0
1
2
3
4
finite element results
limit analysis upper bound: 4.045

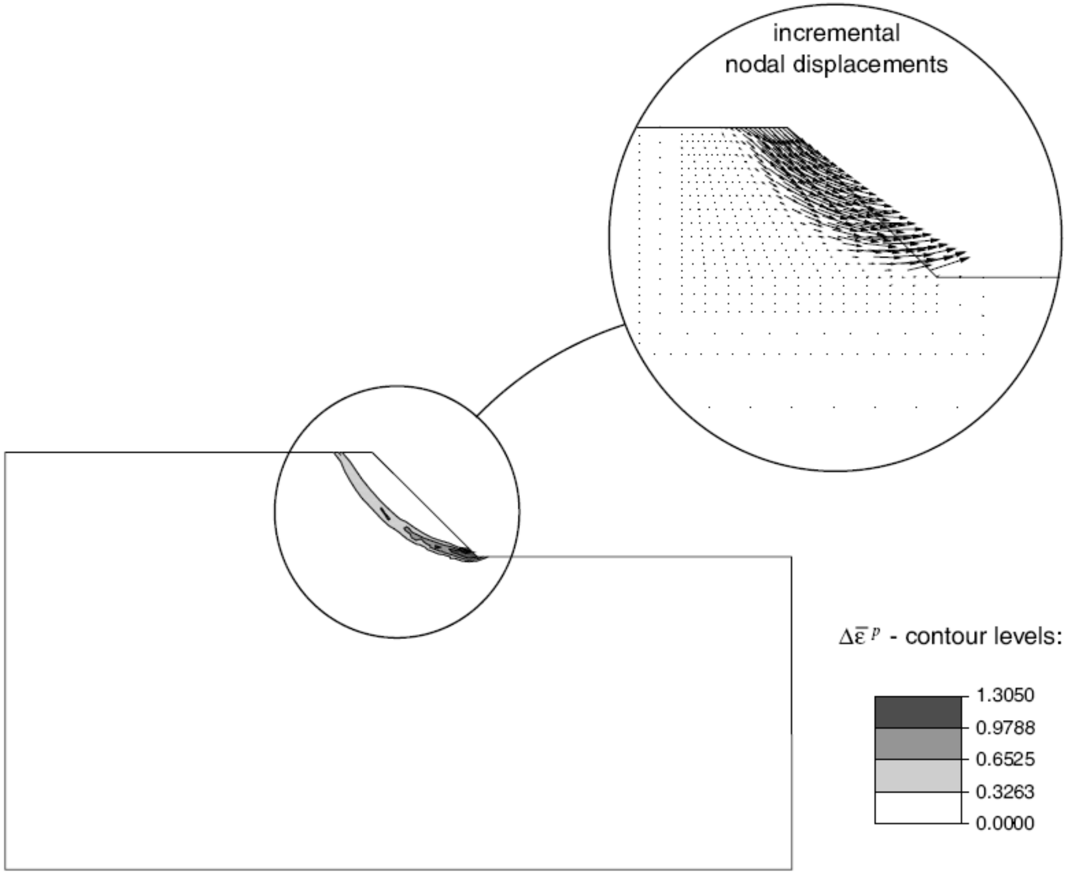
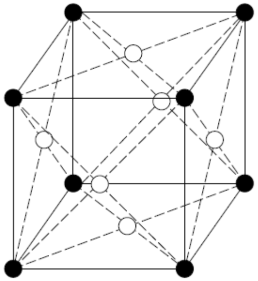


Figure 8.33. Slope stability. Increment of accumulated plastic strain and displacement at collapse (increment 6).

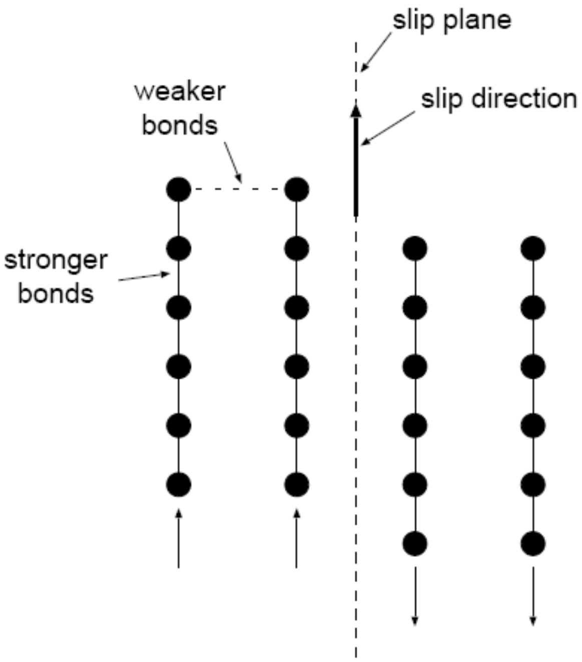
Finite Strain
Single Crystal Plasticity

Slip systems. Plastic deformation by slip

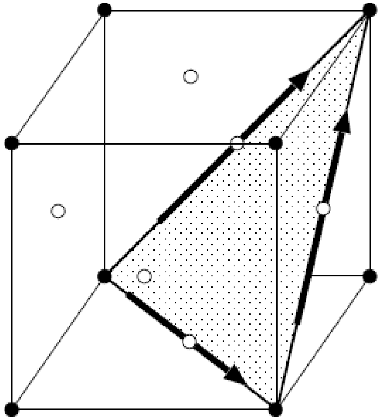
FCC (face-centred cubic) lattice



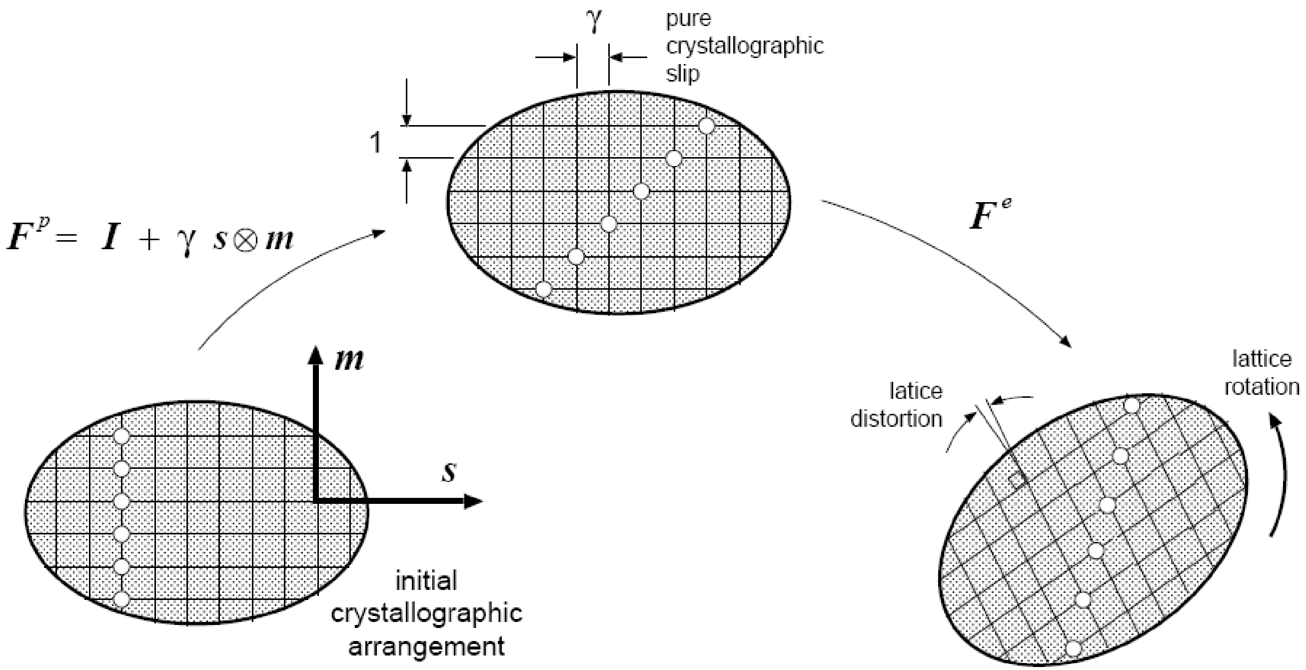
plastic slip mechanism



slip plane and slip directions



Single slip



$$F = F^e\, F^p$$

$$F^p = I + \gamma\ s \otimes m$$

Multi-slip (multi-surface) model

Hyperelastic law

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{F}^e) = \bar{\rho}\, \frac{\partial \psi^e}{\partial \boldsymbol{F}^e}\, \boldsymbol{F}^{e\,T}$$

Plastic flow rule

$$\boldsymbol{L}^p \equiv \dot{\boldsymbol{F}}^p\, (\boldsymbol{F}^p)^{-1} = \sum_{\alpha=1}^{n_{\mathrm{act}}} \dot{\gamma}^\alpha\, \boldsymbol{s}^\alpha \otimes \boldsymbol{m}^\alpha$$

Yield functions

$$\Phi^\alpha(\tau^\alpha, \tau_y^\alpha) \equiv \tau^\alpha - \tau_y^\alpha \qquad \alpha = 1, \ldots, 2n_{\mathrm{syst}}$$

Schmid resolved
(Kirchhoff) shear stress

$$\tau^\alpha \equiv \boldsymbol{R}^{e\,T}\, \boldsymbol{\tau} \boldsymbol{R}^e : \boldsymbol{s}^\alpha \otimes \boldsymbol{m}^\alpha$$

Yield surfaces

$$\Phi^\alpha(\tau^\alpha(\boldsymbol{\tau}), \tau_y^\alpha) = 0$$

Plastic multipliers:

- Rate-independent case (loading/unloading condition)

$$\Phi^{\alpha} \leq 0 \qquad \dot{\gamma}^{\alpha} \geq 0 \qquad \dot{\gamma}^{\alpha}\Phi^{\alpha} = 0 \qquad \alpha = 1, \ldots, 2n_{\mathrm{syst}}$$

- Rate-dependent. For example, Norton type visco-plastic:

$$\dot{\gamma}^{\alpha} = \dot{\gamma}_0 \left(\frac{|\tau^{\alpha}|}{g} \right)^{\frac{1}{M}} \mathrm{sign}(\tau^{\alpha}) \,, \qquad \alpha = 1, \ldots, 2n_{\mathrm{syst}}$$

Backward integration algorithm

Elastic trial step

$$\boldsymbol{F}^{e\ \mathrm{trial}}_{n+1} = \boldsymbol{F}_{\Delta}\,\boldsymbol{F}^{e}_{n}$$

$$\boldsymbol{\tau}^{\mathrm{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi^{e}}{\partial\boldsymbol{F}^{e}}\bigg|^{\mathrm{trial}}_{n+1}\ \boldsymbol{F}^{e\ \mathrm{trial}T}$$

$$\tau^{\alpha\ \mathrm{trial}}_{n+1} = (\boldsymbol{R}^{e\ \mathrm{trial}}_{n+1})^{T}\ \boldsymbol{\tau}^{\mathrm{trial}}_{n+1}\,\boldsymbol{R}^{e\ \mathrm{trial}}_{n+1} : (\boldsymbol{s}^{\alpha}\otimes\boldsymbol{m}^{\alpha})$$

Consistency check

IF $\Phi^{\alpha\ \mathrm{trial}} \equiv \tau^{\alpha\ \mathrm{trial}}_{n+1} - \tau_{y}(\gamma^{\mathrm{trial}}_{n+1}) \leq 0$ THEN $(\cdot)_{n+1} := (\cdot)^{\mathrm{trial}}_{n+1}$

ELSE perform multi-surface return mapping

Multi-vector return mapping

Exponential integrator for the plastic flow

$$\boldsymbol{F}^p_{n+1} = \exp\left[\sum_{\alpha\in\mathscr{A}} \Delta\gamma^\alpha\, \boldsymbol{s}^\alpha \otimes \boldsymbol{m}^\alpha\right] \boldsymbol{F}^p_n \longrightarrow$$

plastic
incompressibility
preserved exactly

$\mathscr{A}$ is the set of n_{act} active slip systems

$$\Downarrow$$

$$\boldsymbol{F}^e_{n+1} = \boldsymbol{F}^{e\ \text{trial}}_{n+1} \exp\left[-\sum_{\alpha\in\mathscr{A}} \Delta\gamma^\alpha\, \boldsymbol{s}^\alpha \otimes \boldsymbol{m}^\alpha\right]$$

Note: $\exp[\boldsymbol{X}] = \displaystyle\sum_{n=0}^{n_{\max}} \frac{1}{n!}\, \boldsymbol{X}^n$

- Rate-independent case:

Consistency equations (1 equation for each active slip system)

$$\tilde{\Phi}^\alpha(\Delta\boldsymbol{\gamma}) \equiv \tau^\alpha(\boldsymbol{F}^e_{n+1}(\Delta\boldsymbol{\gamma})) - \tau_y(\gamma_{n+1}(\Delta\boldsymbol{\gamma})) = 0 \qquad \forall\ \alpha \in \mathscr{A}.$$

Newton-Raphson solution

$$\sum_{\beta\in\mathscr{A}} \mathrm{J}^{\alpha\beta}\,\delta\gamma^\beta = -\tilde{\Phi}^\alpha$$

$$\mathrm{J}^{\alpha\beta} \equiv \frac{\mathrm{d}\tilde{\phi}^\alpha}{\mathrm{d}\Delta\gamma^\beta} = \frac{\mathrm{d}\tau^\alpha}{\mathrm{d}\boldsymbol{F}^e} : \frac{\mathrm{d}\boldsymbol{F}^e}{\mathrm{d}\Delta\gamma^\beta} - H$$

$$\left[\frac{\mathrm{d}\boldsymbol{F}^e}{\mathrm{d}\Delta\gamma^\beta}\right]_{ij} = -F^{e\ \mathrm{trial}}_{im}\,\mathsf{E}_{mjkl}\,[\boldsymbol{s}^\beta \otimes \boldsymbol{m}^\beta]_{kl}$$

$$\Delta\gamma^\alpha_{(k)} := \Delta\gamma^\alpha_{(k-1)} + \delta\gamma^\alpha$$

tensor exponential derivative

- Rate-dependent case (system of six equations in 3-D, regardless of the number of slip systems):

$$\boldsymbol{S}(\boldsymbol{F}^e) \equiv \boldsymbol{F}^e - \boldsymbol{F}^{e\text{ trial}} \exp\left[-\Delta t \sum_{\alpha=1}^{n_{\text{syst}}} \dot{\gamma}^\alpha(\boldsymbol{F}^e)\, \boldsymbol{s}^\alpha \otimes \boldsymbol{m}^\alpha\right] = \boldsymbol{0}$$

System Jacobian for N-R algorithm:

$$\mathsf{J}_{ijkl} \equiv \left[\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}\boldsymbol{F}^e}\right]_{ijkl} = \delta_{ik}\delta_{jl} + \Delta t\; F^{e\text{ trial}}_{im}\, \mathsf{E}_{mjpq} \left[\sum_{\alpha=1}^{n_{\text{syst}}} \boldsymbol{s}_0^\alpha \otimes \boldsymbol{m}_0^\alpha \otimes \frac{\mathrm{d}\dot{\gamma}^\alpha}{\mathrm{d}\boldsymbol{F}^e}\right]_{pqkl}$$

Consistent elasto-plastic spatial tangent modulus

- Rate-independent case (assuming neo-Hookean elastic law)

$$\boxed{\mathsf{a}^{ep} = \mathsf{a}^{e} + \mathsf{a}^{p}}$$

$$\mathsf{a}^{p} \equiv -\frac{2G^2}{J}\, \mathsf{d} : \mathsf{e} : \mathsf{v}$$

$$\mathsf{e}_{ijkl} \equiv \left(\boldsymbol{F}^{e}_{\mathrm{iso}}\right)_{il} \left(\boldsymbol{F}^{e\,\mathrm{trial}}_{\mathrm{iso}}\right)_{jk}$$

$$\mathsf{v}_{ijkl} \equiv [\mathsf{E} : \mathsf{U}]_{ijkm} \left(\boldsymbol{F}^{e}_{\mathrm{iso}}\right)_{lm} - \frac{1}{3}[\mathsf{E} : \mathsf{U} : \boldsymbol{F}^{e}_{\mathrm{iso}}]_{ij}\, \delta_{kl}$$

$$\mathsf{U} \equiv \sum_{\alpha \in \mathscr{A}} \sum_{\beta \in \mathscr{A}} [\mathbf{J}^{-1}]^{\alpha\beta}\, \boldsymbol{s}^{\alpha} \otimes \boldsymbol{m}^{\alpha} \otimes [\bar{\boldsymbol{s}}^{\beta} \otimes \boldsymbol{m}^{\beta} + \bar{\boldsymbol{m}}^{\beta} \otimes \boldsymbol{s}^{\beta}]$$

- Rate-dependent case (assuming neo-Hookean elastic law)

$$\boxed{\mathsf{a}_{ijkl} = \left[\tfrac{G}{J^{4/3}}\mathsf{F} : \mathsf{A} \ : \mathsf{H} + \tfrac{K}{J}\boldsymbol{I} \otimes \boldsymbol{I}\right]_{ijkl} - \sigma_{il}\delta_{jk}}$$

$$\mathsf{F}_{ijkl} \equiv \delta_{ik}(\boldsymbol{F}^{e}_{\mathrm{iso}})_{jl} + \delta_{jk}(\boldsymbol{F}^{e}_{\mathrm{iso}})_{il} - \tfrac{1}{3}\delta_{ij}(\boldsymbol{F}^{e}_{\mathrm{iso}})_{kl}$$

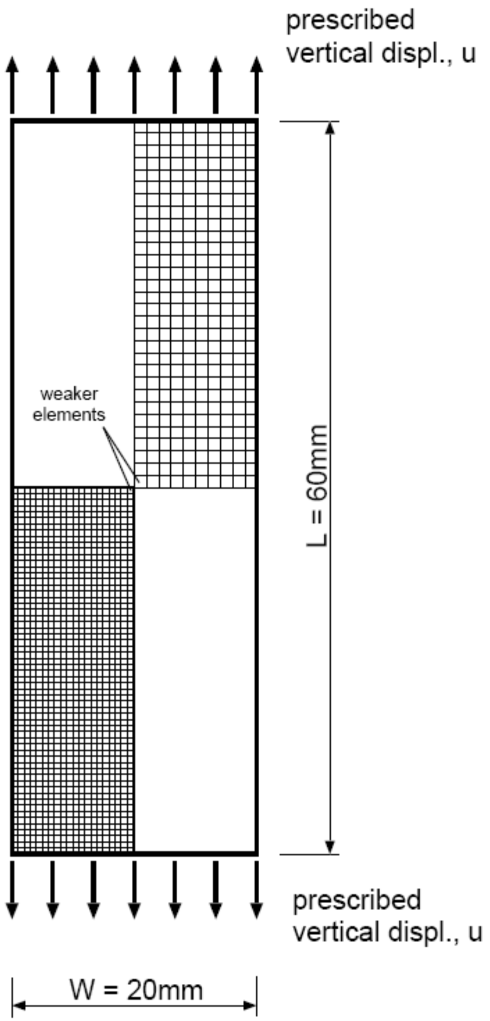
$$\mathsf{H}_{ijkl} \equiv \delta_{ik}F_{lj} - \tfrac{1}{3}F_{ij}\delta_{kl} \ .$$

$$\mathsf{A} \equiv \frac{\partial \boldsymbol{F}^{e}_{\mathrm{iso}}}{\partial \boldsymbol{F}_{\mathrm{iso}}} \longrightarrow \text{only algorithm-related term}$$

$$\mathsf{A} = \mathsf{J}^{-1} : \mathsf{C}$$

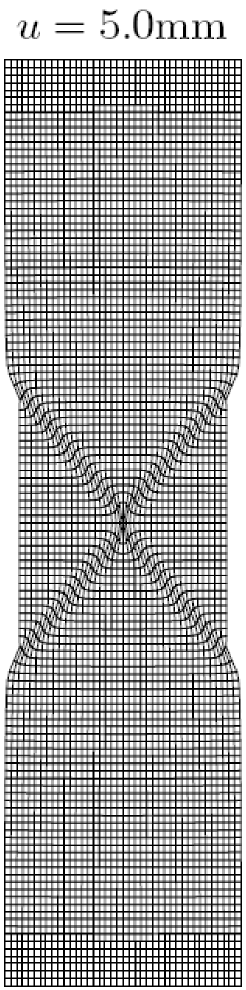
$$\mathsf{C}_{ijkl} \equiv -\delta_{ik}\,(\boldsymbol{F}^{p\ -1}_{n+1})_{lj}$$

Example. Strain localisation (rate-independent)



Planar double-slip crystal models
(plane strain)

Slip planes symmetric about longitudinal axis



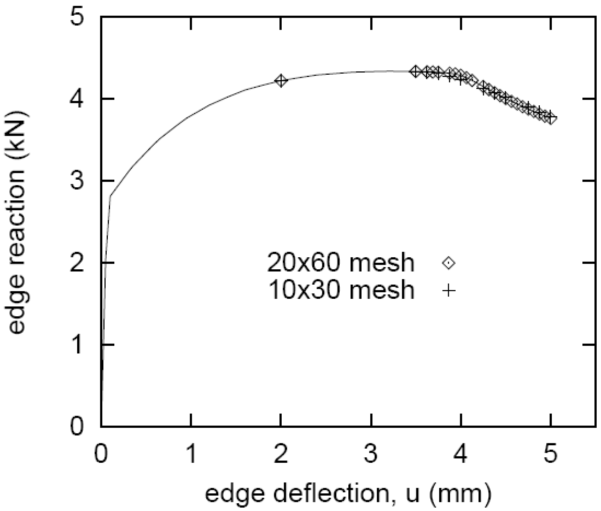
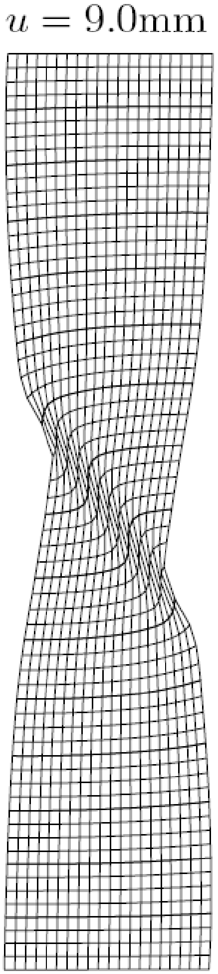
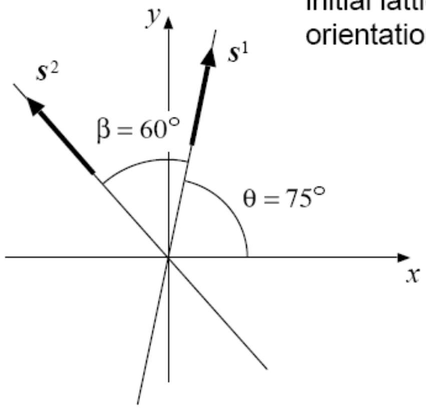


Table 16.4: Rectangular strip. Convergence table.

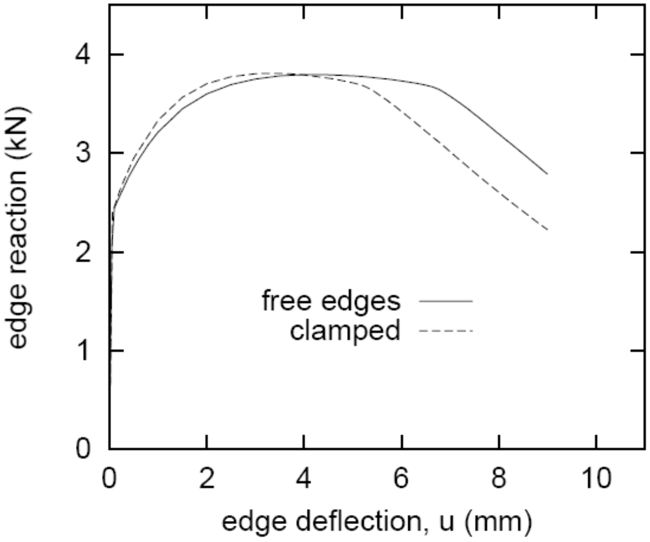
	relative residual norm (%) – 20 × 60 mesh		
iter. no.	incr. 6	incr. 13	incr. 20
1	$0.201249E{+}01$	$0.210379E{+}02$	$0.187277E{+}02$
2	$0.543621E{+}01$	$0.138596E{+}01$	$0.889781E{+}01$
3	$0.129385E{+}01$	$0.171953E{-}00$	$0.291015E{-}01$
4	$0.106128E{+}00$	$0.142736E{-}01$	$0.171479E{-}05$
5	$0.592669E{-}01$	$0.645938E{-}04$	$0.444649E{-}10$
6	$0.603329E{-}01$	$0.212211E{-}09$	
7	$0.693394E{-}02$		
8	$0.367075E{-}03$		
9	$0.916724E{-}09$		

Slip planes *unsymmetric* about the longitudinal axis





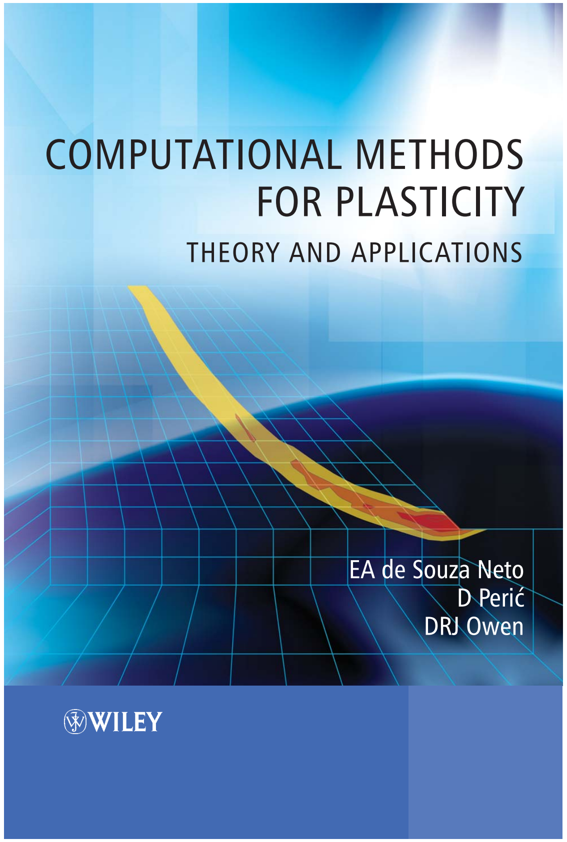
initial lattice orientation



Freely downloadable FORTRAN FE program available at:

www.wiley.com/go/desouzaneto

The companion software to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