# *Finite Strain Plasticity*

***Topics:***

- *1-D Model with multiplicative kinematics*
- *General isotropic 3-D model*
- *Exponential map-based integration algorithm*
- *Consistent tangent operator*

# 1-D Model with Multiplicative Kinematics

# 1D Multiplicative Kinematics. Justification

Axial stretch

$$\lambda = \frac{l}{l_0}$$

Plastic axial stretch & unstressed configuration

$$\lambda^p = \frac{l_p}{l_0}$$

Elastic axial stretch

$$\lambda^e = \frac{l}{l_p}$$

Then, the multiplicative split

$$\boxed{\lambda = \lambda^e \lambda^p}$$

follows **naturally**.

# 1D Hencky Hyperelastic Law

$$\tau = E\,\varepsilon^e, \qquad \varepsilon^e = \ln \lambda^e$$

# Yield function & elastic domain

$$\Phi(\tau, \tau_y) = |\tau| - \tau_y$$

$$\mathscr{E} = \{\tau \mid \Phi(\tau, \tau_y) < 0\}$$

# Flow Rule

In addition to the above, a plastic flow rule is required to define the evolution of the plastic stretch, $\lambda^p$. Here we shall postulate the following law

$$\dot{\lambda}^p(\lambda^p)^{-1} = \dot{\gamma}\,\mathrm{sign}(\tau), \tag{14.10}$$

which is complemented by the usual condition

$$\Phi \leq 0, \quad \dot{\gamma} \geq 0, \quad \dot{\gamma}\Phi = 0. \tag{14.11}$$

Note that, from the definition of $\lambda^p$, we have the identity

$$\dot{\lambda}^p(\lambda^p)^{-1} = \frac{\dot{l}_p}{l_p}, \tag{14.12}$$

that is, the right-hand side of (14.10) is the instantaneous rate of plastic straining measured with respect to the plastic configuration defined by $\lambda^p$. Also note, with the *logarithmic axial plastic strain* defined by

$$\varepsilon^p = \ln \lambda^p, \tag{14.13}$$

that (14.10) can be equivalently expressed as

$$\dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\tau), \tag{14.14}$$

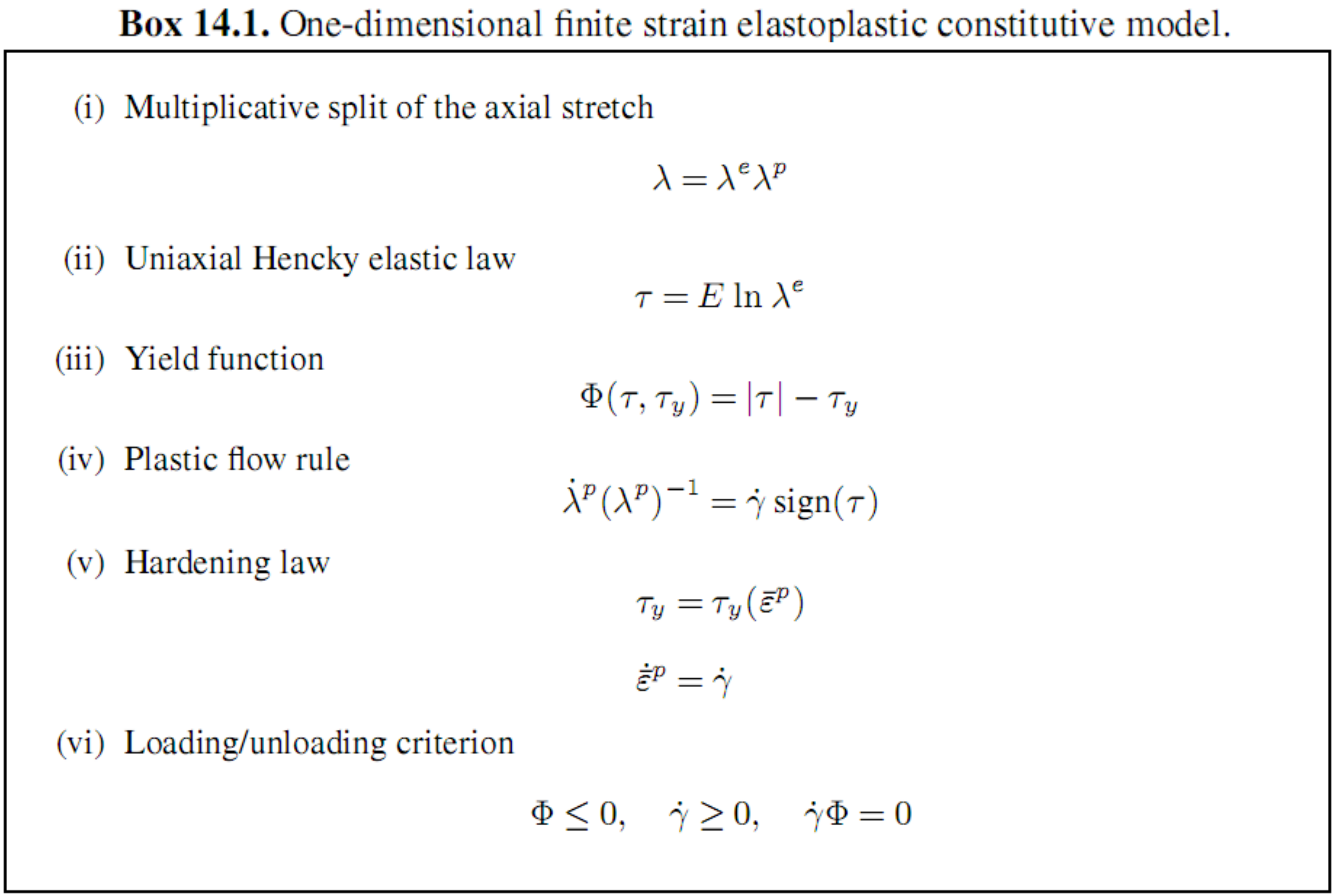
which has the same functional format as its infinitesimal counterpart (6.10). We remark, however, that the above law in terms of the logarithmic axial plastic strain does not generalise to the multiaxial case treated in Section 14.3.

# Hardening

$$\tau_y = \tau_y(\bar{\varepsilon}^p)$$

$$\dot{\bar{\varepsilon}}^p = |\dot{\varepsilon}^p| = \dot{\gamma}$$

$$\dot{\gamma} = \frac{E}{H+E}\,\mathrm{sign}(\tau)\,\frac{\dot{\lambda}}{\lambda} \qquad \dot{\gamma} = \frac{E}{H+E}\,|\dot{\varepsilon}|$$



**Box 14.1.** One-dimensional finite strain elastoplastic constitutive model.

(i) Multiplicative split of the axial stretch

$$\lambda = \lambda^e \lambda^p$$

(ii) Uniaxial Hencky elastic law

$$\tau = E \ln \lambda^e$$

(iii) Yield function

$$\Phi(\tau, \tau_y) = |\tau| - \tau_y$$

(iv) Plastic flow rule

$$\dot{\lambda}^p (\lambda^p)^{-1} = \dot{\gamma}\, \text{sign}(\tau)$$

(v) Hardening law

$$\tau_y = \tau_y(\bar{\varepsilon}^p)$$

$$\dot{\bar{\varepsilon}}^p = \dot{\gamma}$$

(vi) Loading/unloading criterion

$$\Phi \leq 0, \quad \dot{\gamma} \geq 0, \quad \dot{\gamma}\Phi = 0$$

# *General Multi-dimensional Model with Multiplicative Kinematics*

# Multiplicative split of the deformation gradient

$$\boxed{\boldsymbol{F} = \boldsymbol{F}^{e}\boldsymbol{F}^{p}}$$

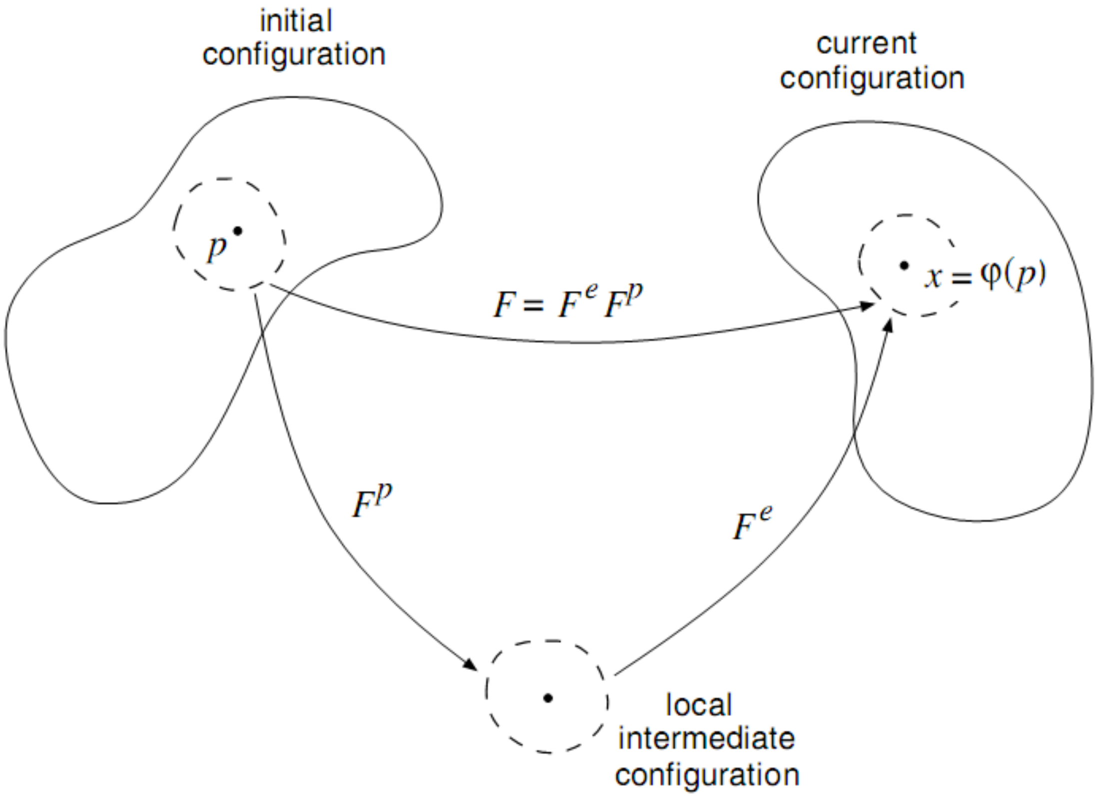


**Figure 14.1.** Multiplicative decomposition of the deformation gradient.

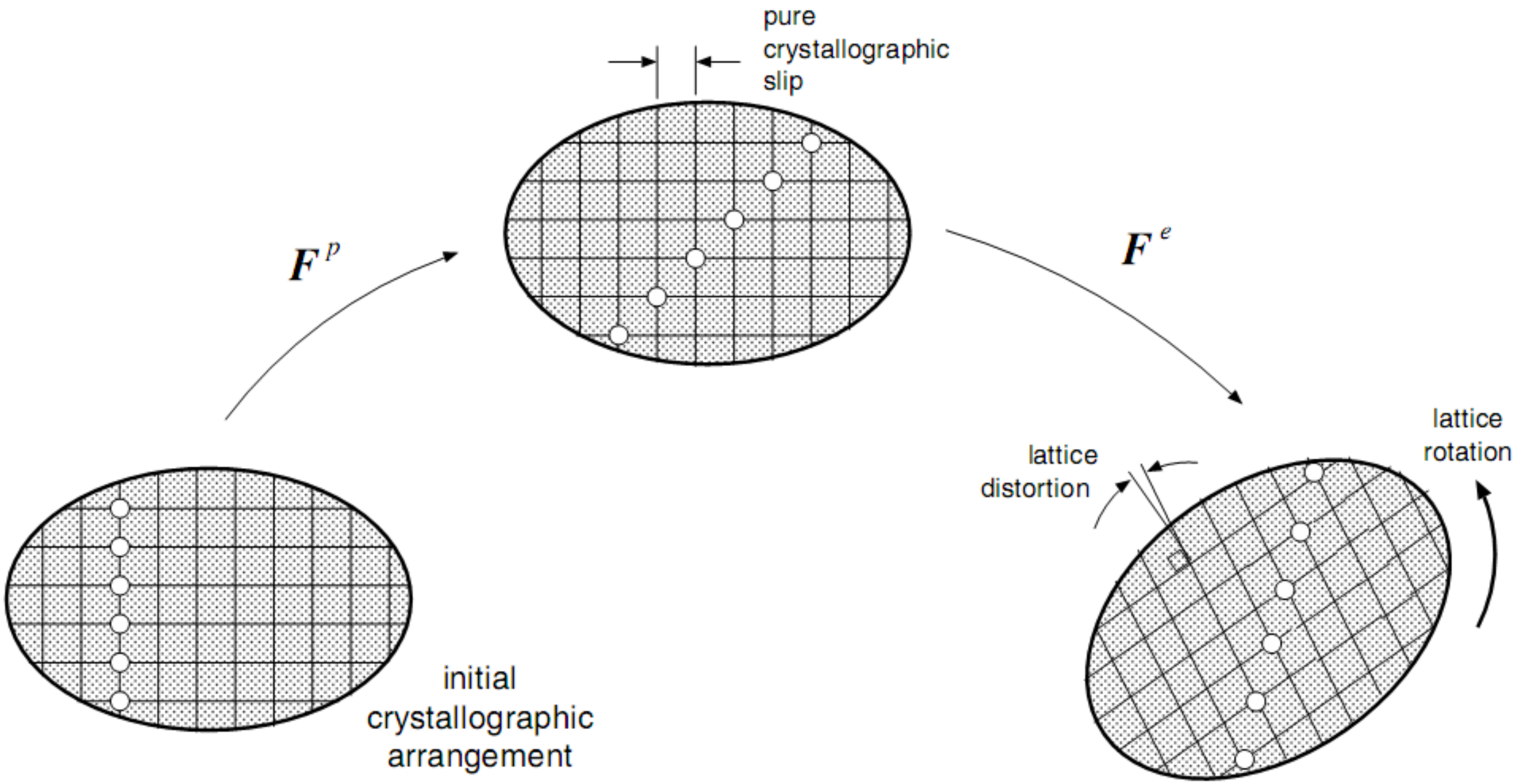


**Figure 14.2.** Multiplicative decomposition of the deformation gradient. Micromechanical basis.

## Elastic and plastic stretches and rotations

$$\boldsymbol{F}^e = \boldsymbol{R}^e\,\boldsymbol{U}^{\,e} = \boldsymbol{V}^e\boldsymbol{R}^e$$

$$\boldsymbol{F}^p = \boldsymbol{R}^p\,\boldsymbol{U}^{\,p} = \boldsymbol{V}^p\boldsymbol{R}^p$$

## Elastic and plastic stretches and rotations

$$\boldsymbol{L} \equiv \nabla_x \boldsymbol{v} = \dot{\boldsymbol{F}}\,\boldsymbol{F}^{\,-1}$$

$$\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{F}^e\boldsymbol{L}^p(\boldsymbol{F}^e)^{-1}$$

$$\boldsymbol{L}^e \equiv \dot{\boldsymbol{F}}^e(\boldsymbol{F}^e)^{-1}, \quad \boldsymbol{L}^p \equiv \dot{\boldsymbol{F}}^p(\boldsymbol{F}^p)^{-1}$$

## Plastic stretching and spin

$$\boldsymbol{D}^p \equiv \mathrm{sym}[\boldsymbol{L}^p], \quad \boldsymbol{W}^p \equiv \mathrm{skew}[\boldsymbol{L}^p]$$

$$\boldsymbol{D}^p = \sum_{i=1}^{3} d_i^p\,\boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

# Spatially rotated plastic stretching

$$\tilde{\boldsymbol{D}}^p \equiv \boldsymbol{R}^e\, \boldsymbol{D}^p\, \boldsymbol{R}^{eT} = \boldsymbol{R}^e\, \mathrm{sym}[\dot{\boldsymbol{F}}^p \boldsymbol{F}^{p\,-1}]\, \boldsymbol{R}^{eT}$$

$$\tilde{\boldsymbol{D}}^p = \sum_{i=1}^{3} d_i^p\, \tilde{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}_i$$

# Logarithmic elastic strain measure

$$\boxed{\boldsymbol{\varepsilon}^e \equiv \ln \boldsymbol{V}^e = \tfrac{1}{2} \ln \boldsymbol{B}^e}$$

$$\boldsymbol{B}^e = \boldsymbol{F}^e (\boldsymbol{F}^e)^T = (\boldsymbol{V}^e)^2$$

Deviatoric-volumetric split

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon}^e_{\mathrm{d}} + \tfrac{1}{3} \varepsilon^e_{\mathrm{v}} \, \boldsymbol{I}$$

$$\boldsymbol{\varepsilon}^e_{\mathrm{d}} \equiv \boldsymbol{\varepsilon}^e - \tfrac{1}{3} \, \mathrm{tr}[\boldsymbol{\varepsilon}^e] \, \boldsymbol{I}$$

$$\varepsilon^e_{\mathrm{v}} \equiv \mathrm{tr}[\boldsymbol{\varepsilon}^e] = \ln J^e$$

# General isotropic large strain elasto-plasticity model

Hyperelasticity law

$$\psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha})$$

$$\boldsymbol{\tau} = \bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}$$

$$A_i = \bar{\rho}\,\frac{\partial \psi}{\partial \alpha_i} \quad (i = 1, \dots, k)$$

Yield function & yield criterion

$$\mathscr{E} = \{\boldsymbol{\tau} \mid \Phi(\boldsymbol{\tau}, \boldsymbol{A}) < 0\}$$

$$\mathscr{Y} = \{\boldsymbol{\tau} \mid \Phi(\boldsymbol{\tau}, \boldsymbol{A}) = 0\}$$

## Plastic flow rule

$$\left.\begin{array}{l} \tilde{\boldsymbol{D}}^p = \dot{\gamma}\,\dfrac{\partial \Psi}{\partial \boldsymbol{\tau}} \\[4ex] \boldsymbol{W}^p = \boldsymbol{0} \end{array}\right\} \Rightarrow \boxed{\boldsymbol{L}^p \equiv \dot{\boldsymbol{F}}^p (\boldsymbol{F}^p)^{-1} = \dot{\gamma}\,(\boldsymbol{R}^e)^T \,\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\, \boldsymbol{R}^e}$$

…recall 1D counterpart

$$\dot{\lambda}^p (\lambda^p)^{-1} = \dot{\gamma}\,\mathrm{sign}(\tau)$$

## Internal variable evolution

$$\dot{\boldsymbol{\alpha}} = \dot{\gamma}\,\boldsymbol{H}(\boldsymbol{\tau}, \boldsymbol{A})$$

$$\Phi \leq 0, \qquad \dot{\gamma} \geq 0, \qquad \dot{\gamma}\Phi = 0$$

**Box 14.2.** General isotropic finite strain multiplicative elastoplastic model.

(i) Multiplicative decomposition of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{F}^{e} \boldsymbol{F}^{p}$$

(ii) Isotropic hyperelastic law

$$\boldsymbol{\tau} = \bar{\rho}\, \frac{\partial \psi(\boldsymbol{\varepsilon}^{e}, \boldsymbol{\alpha})}{\partial \boldsymbol{\varepsilon}^{e}}$$

(iii) Evolution equations for $\boldsymbol{F}^{p}$ and internal variable set $\boldsymbol{\alpha}$

$$\dot{\boldsymbol{F}}^{p} \boldsymbol{F}^{p-1} = \dot{\gamma}\, \boldsymbol{R}^{eT} \frac{\partial \Psi}{\partial \boldsymbol{\tau}} \boldsymbol{R}^{e}$$

$$\dot{\boldsymbol{\alpha}} = \dot{\gamma}\, \boldsymbol{H}$$

(iv) Loading/unloading criterion

$$\Phi \leq 0, \quad \dot{\gamma} \geq 0, \quad \dot{\gamma}\Phi = 0$$

# Some important properties

Dissipation inequality

$$\boldsymbol{\tau} : \tilde{\boldsymbol{D}}^p - \boldsymbol{A} * \dot{\boldsymbol{\alpha}} \geq 0$$

Volume-preserving plastic deformations

$$J^p \equiv \det \boldsymbol{F}^p$$

$$\begin{aligned} \varepsilon_{\mathrm{v}}^p \equiv \ln J^p &= \ln[\lambda_{(1)}^p \, \lambda_{(2)}^p \, \lambda_{(3)}^p] \\ &= \ln \lambda_{(1)}^p + \ln \lambda_{(2)}^p + \ln \lambda_{(3)}^p = \mathrm{tr}[\ln \boldsymbol{V}^p] \end{aligned}$$

Logarithmic
volumetric
plastic strain

$$\boxed{\det \boldsymbol{F}^p = 1 \quad \Longleftrightarrow \quad \varepsilon_{\mathrm{v}}^p = 0}$$

# Plastic incompressibility at finite strains

$$\begin{aligned}\dot{\varepsilon}^p_{\mathrm{v}} \equiv (\ln[\det \boldsymbol{F}^p])^{\cdot} &= \frac{1}{\det \boldsymbol{F}^p}(\det \boldsymbol{F}^p)^{\cdot}\\ &= \mathrm{tr}[\dot{\boldsymbol{F}}^{p}(\boldsymbol{F}^p)^{-1}].\end{aligned}$$

$$\dot{\varepsilon}^p_{\mathrm{v}} = \dot{\gamma}\ \mathrm{tr}\left[\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right]$$

$$\boxed{\dot{\varepsilon}^p_{\mathrm{v}} = 0 \quad \Longleftrightarrow \quad \mathrm{tr}\left[\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right] = 0}$$

Analogous to the small strain theory!!

# *General Elastic Predictor/Return-mapping Algorithm*

**Problem 14.1 (The finite elastoplasticity initial value problem).** *Given the initial values* $\boldsymbol{F}^p(t_0)$ *and* $\boldsymbol{\alpha}(t_0)$ *and given the history of the deformation gradient* $\boldsymbol{F}(t)$, $t \in [t_0, T]$, *find the functions* $\boldsymbol{F}^p(t)$, $\boldsymbol{\alpha}(t)$ *and* $\dot{\gamma}(t)$ *that satisfy*

$$\begin{aligned} \dot{\boldsymbol{F}}^p(t)\,[\boldsymbol{F}^p(t)]^{-1} &= \dot{\gamma}(t)\,\boldsymbol{R}^e(t)^T \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_t \boldsymbol{R}^e(t) \\ \dot{\boldsymbol{\alpha}}(t) &= \dot{\gamma}(t)\,\boldsymbol{H}(\boldsymbol{\tau}(t), \boldsymbol{A}(t)) \end{aligned} \tag{14.66}$$

*and*

$$\dot{\gamma}(t) \geq 0, \quad \Phi(\boldsymbol{\tau}(t), \boldsymbol{A}(t)) \leq 0, \quad \dot{\gamma}(t)\Phi(\boldsymbol{\tau}(t), \boldsymbol{A}(t)) = 0 \tag{14.67}$$

*for each instant* $t \in [t_0, T]$, *with*

$$\boldsymbol{\tau}(t) = \bar{\rho}\left.\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\right|_t, \quad \boldsymbol{A}(t) = \bar{\rho}\left.\frac{\partial \psi}{\partial \boldsymbol{\alpha}}\right|_t \tag{14.68}$$

*and the kinematic relations*

$$\begin{aligned} \boldsymbol{\varepsilon}^e(t) &= \ln \boldsymbol{V}^e(t) \\ \boldsymbol{V}^e(t) &= [\boldsymbol{F}^e(t)\,\boldsymbol{F}^e(t)^T]^{\frac{1}{2}} \\ \boldsymbol{R}^e(t) &= [\boldsymbol{V}^e(t)]^{-1}\,\boldsymbol{F}^e(t) \\ \boldsymbol{F}^e(t) &= \boldsymbol{F}(t)\,[\boldsymbol{F}^p(t)]^{-1}. \end{aligned} \tag{14.69}$$

# Plastic flow discretisation. Exponential map backward scheme

$$\boldsymbol{F}^p_{n+1} = \exp\left[\Delta\gamma\, \boldsymbol{R}^{e\,T}_{n+1} \left.\frac{\partial\Psi}{\partial\boldsymbol{\tau}}\right|_{n+1} \boldsymbol{R}^e_{n+1}\right] \boldsymbol{F}^p_n$$

$$= \boldsymbol{R}^{e\,T}_{n+1} \exp\left[\Delta\gamma \left.\frac{\partial\Psi}{\partial\boldsymbol{\tau}}\right|_{n+1}\right] \boldsymbol{R}^e_{n+1}\, \boldsymbol{F}^p_n$$

The exponential map algorithm is particularly suitable for the present plastic flow equation

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + \Delta\gamma\, \boldsymbol{H}_{n+1}$$

$$\Delta\gamma \geq 0, \quad \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \leq 0, \quad \Delta\gamma\, \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) = 0$$

**Important property. Exact volume-preserving discretised flow**

$$\boxed{\operatorname{tr}\left[\frac{\partial\Psi}{\partial\boldsymbol{\tau}}\right] = 0 \quad \Longleftrightarrow \quad \det[\exp[\Delta\gamma\, \partial\Psi/\partial\boldsymbol{\tau}]] = 1}$$

# Equivalent elastic deformation gradient update

$$\boldsymbol{F}^e_{n+1} = \boldsymbol{F}_\Delta \boldsymbol{F}^e_n \, \boldsymbol{R}^{e\,T}_{n+1} \exp\left[ -\Delta\gamma \left. \frac{\partial \Psi}{\partial \boldsymbol{\tau}} \right|_{n+1} \right] \boldsymbol{R}^e_{n+1}$$

**Problem 14.2 (The incremental finite plasticity problem).** *Given* $\boldsymbol{F}^e_n$ *and* $\boldsymbol{\alpha}_n$ *at the beginning of the interval* $[t_n, t_{n+1}]$ *and given the prescribed incremental deformation gradient,* $\boldsymbol{F}_\Delta$*, for this interval, solve the following system of algebraic equations*

$$\begin{aligned} \boldsymbol{F}^e_{n+1} &= \boldsymbol{F}_\Delta \boldsymbol{F}^e_n\, \boldsymbol{R}^{e\,T}_{n+1}\, \exp\left[ -\Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \right] \boldsymbol{R}^e_{n+1} \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\gamma\, \boldsymbol{H}_{n+1} \end{aligned} \tag{14.77}$$

*for the unknowns* $\boldsymbol{F}^e_{n+1}$, $\boldsymbol{\alpha}_{n+1}$ *and* $\Delta\gamma$*, subjected to the constraints*

$$\Delta\gamma \geq 0, \quad \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \leq 0, \quad \Delta\gamma\, \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) = 0, \tag{14.78}$$

*with*

$$\boldsymbol{\tau}_{n+1} = \bar{\rho} \left.\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\right|_{n+1}, \quad \boldsymbol{A}_{n+1} = \bar{\rho} \left.\frac{\partial \psi}{\partial \boldsymbol{\alpha}}\right|_{n+1} \tag{14.79}$$

*and the kinematic relations*

$$\begin{aligned} \boldsymbol{\varepsilon}^e_{n+1} &= \ln \boldsymbol{V}^e_{n+1} \\ \boldsymbol{V}^e_{n+1} &= [\boldsymbol{F}^e_{n+1}\, (\boldsymbol{F}^e_{n+1})^T]^{\frac{1}{2}} \\ \boldsymbol{R}^e_{n+1} &= [\boldsymbol{V}^e_{n+1}]^{-1}\, \boldsymbol{F}^e_{n+1}. \end{aligned} \tag{14.80}$$

# Logarithmic elastic strain-based update. Small strain format

$$\boldsymbol{V}^{e}_{n+1} = \boldsymbol{F}^{e\,\mathrm{trial}}_{n+1}\, \boldsymbol{R}^{e\,T}_{n+1} \exp\left[ -\Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \right]$$

$$\boldsymbol{V}^{e}_{n+1} \exp\left[ \Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \right] = \boldsymbol{F}^{e\,\mathrm{trial}}_{n+1}\, \boldsymbol{R}^{e\,T}_{n+1}$$

$$\boldsymbol{V}^{e}_{n+1} \exp\left[ 2\,\Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \right] \boldsymbol{V}^{e}_{n+1} = (\boldsymbol{V}^{e\,\mathrm{trial}}_{n+1})^2$$

$$\boldsymbol{V}^{e}_{n+1} = \boldsymbol{V}^{e\,\mathrm{trial}}_{n+1} \exp\left[ -\Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \right]$$

$$\boxed{\boldsymbol{\varepsilon}^{e}_{n+1} = \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1} - \Delta\gamma \left.\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1}}$$

**Small strain format !!!**

**Box 14.3.** General integration algorithm for isotropic multiplicative finite strain elastoplasticity.

HYPLAS procedure: `MATISU`

(i) Given incremental displacement $\Delta \boldsymbol{u}$, update the deformation gradient

$$\boldsymbol{F}_{\Delta} := \boldsymbol{I} + \nabla_n[\Delta \boldsymbol{u}], \quad \boldsymbol{F}_{n+1} := \boldsymbol{F}_{\Delta}\, \boldsymbol{F}_n$$

(ii) Compute elastic trial state

$$\begin{aligned} \boldsymbol{B}^e_n &:= \exp[2\, \boldsymbol{\varepsilon}^e_n] \\ \boldsymbol{B}^{e\,\text{trial}}_{n+1} &:= \boldsymbol{F}_{\Delta}\, \boldsymbol{B}^e_n\, (\boldsymbol{F}_{\Delta})^T \\ \boldsymbol{\varepsilon}^{e\,\text{trial}}_{n+1} &:= \ln[\boldsymbol{V}^{e\,\text{trial}}_{n+1}] = \tfrac{1}{2} \ln[\boldsymbol{B}^{e\,\text{trial}}_{n+1}] \\ \boldsymbol{\alpha}^{\text{trial}}_{n+1} &:= \boldsymbol{\alpha}_n \end{aligned}$$

(iii) **GOTO BOX 14.4** – small-strain algorithm (update $\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\alpha}$)

(iv) Update the Cauchy stress

$$\boldsymbol{\sigma}_{n+1} := \det[\boldsymbol{F}_{n+1}]^{-1}\, \boldsymbol{\tau}_{n+1}$$

**Box 14.4.** General integration procedure – small strains.

(i) Given $\varepsilon_{n+1}^{e\,\mathrm{trial}}$ and $\boldsymbol{\alpha}_{n+1}^{\mathrm{trial}}$, compute

$$\boldsymbol{\tau}_{n+1}^{\mathrm{trial}} = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\bigg|_{n+1}^{\mathrm{trial}}, \qquad \boldsymbol{A}_{n+1}^{\mathrm{trial}} = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\alpha}}\bigg|_{n+1}^{\mathrm{trial}}$$

(ii) Check plastic admissibility

$$\text{IF } \Phi(\boldsymbol{\tau}_{n+1}^{\mathrm{trial}}, \boldsymbol{A}_{n+1}^{\mathrm{trial}}) \leq 0$$

$$\text{THEN set} \quad (\cdot)_{n+1} := (\cdot)_{n+1}^{\mathrm{trial}} \quad \text{and EXIT}$$

(iii) Return mapping. Solve the algebraic system

$$\left\{ \begin{array}{l} \boldsymbol{\varepsilon}_{n+1}^{e} - \boldsymbol{\varepsilon}_{n+1}^{e\,\mathrm{trial}} + \Delta\gamma\, \dfrac{\partial \Psi}{\partial \boldsymbol{\tau}}\bigg|_{n+1} \\ \boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_{n} - \Delta\gamma\, \boldsymbol{H}(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \\ \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ 0 \end{array} \right\}$$

for $\boldsymbol{\varepsilon}_{n+1}^{e}$, $\boldsymbol{\alpha}_{n+1}$ and $\Delta\gamma$, with

$$\boldsymbol{\tau}_{n+1} = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\bigg|_{n+1}, \qquad \boldsymbol{A}_{n+1} = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\alpha}}\bigg|_{n+1}$$

(iv) EXIT

# *General*

# *Consistent Spatial Tangent Modulus*

## General formula

$$\mathsf{a}_{ijkl} = \frac{1}{J}\,\frac{\partial \tau_{ij}}{\partial F_{kq}}\,F_{lq} - \sigma_{il}\,\delta_{jk}$$

## Algorithmic constitutive function

$$\boldsymbol{\tau}_{n+1} = \tilde{\boldsymbol{\tau}}\,(\boldsymbol{\alpha}_n,\, \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1})$$

$$\hat{\boldsymbol{\tau}}\,(\boldsymbol{\alpha}_n,\, \boldsymbol{F}_{n+1}) = \tilde{\boldsymbol{\tau}}\,(\boldsymbol{\alpha}_n,\, \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1}(\boldsymbol{B}^{e\,\mathrm{trial}}_{n+1}(\boldsymbol{F}^{p}_{n},\, \boldsymbol{F}_{n+1})))$$

## Consistent stress derivative

$$\frac{\partial \hat{\boldsymbol{\tau}}}{\partial \boldsymbol{F}_{n+1}} = \underbrace{\frac{\partial \tilde{\boldsymbol{\tau}}}{\partial \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1}}}_{\text{small strain format}} : \overbrace{\frac{\partial \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1}}{\partial \boldsymbol{B}^{e\,\mathrm{trial}}_{n+1}} : \frac{\partial \boldsymbol{B}^{e\,\mathrm{trial}}_{n+1}}{\partial \boldsymbol{F}_{n+1}}}^{\text{kinematics only}}$$

# Final format

$$\boxed{\mathsf{a}_{ijkl} = \frac{1}{2\,J}\,[\,\mathsf{D} : \mathsf{L} : \mathsf{B}\,]_{ijkl} - \sigma_{il}\,\delta_{jk}}$$

$$\mathsf{D} = \frac{\partial \tilde{\boldsymbol{\tau}}}{\partial \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}}$$

$$\mathsf{L} = \frac{\partial \ln[\boldsymbol{B}^{e\ \mathrm{trial}}_{n+1}]}{\partial \boldsymbol{B}^{e\ \mathrm{trial}}_{n+1}}$$

$$\mathsf{B}_{ijkl} = \delta_{ik}\,(\boldsymbol{B}^{e\ \mathrm{trial}}_{n+1})_{jl} + \delta_{jk}\,(\boldsymbol{B}^{e\ \mathrm{trial}}_{n+1})_{il}$$

# Quadratic rates of asymptotic convergence.

## Example 14.9.1 Tresca bar