Introduction to Finite Strains

Topics:

- *Finite strain kinematics*
- *Stress measures*
- *Hyperelasticity*

Finite Strain Kinematics

The deformation map

$$\varphi : \Omega \longrightarrow \mathscr{E}$$

$$x = \varphi(p)$$

$$x = p + u(p)$$

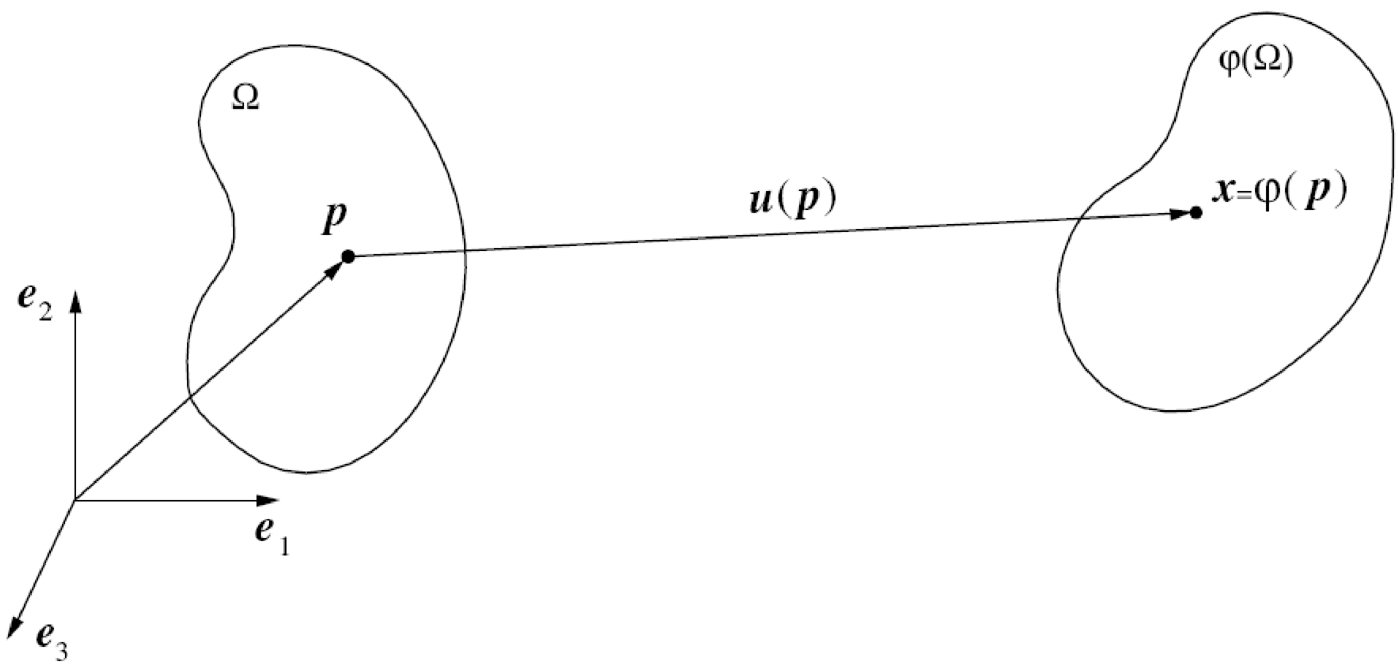


Figure 3.1. Deformation.

Rigid deformation

$$\varphi(\boldsymbol{p}) = \varphi(\boldsymbol{q}) + \boldsymbol{R}\,(\boldsymbol{p} - \boldsymbol{q})$$

$$\varphi(\boldsymbol{p}) = \boldsymbol{p} + \boldsymbol{u}$$

$$\varphi(\boldsymbol{p}) = \boldsymbol{q} + \boldsymbol{R}\,(\boldsymbol{p} - \boldsymbol{q})$$

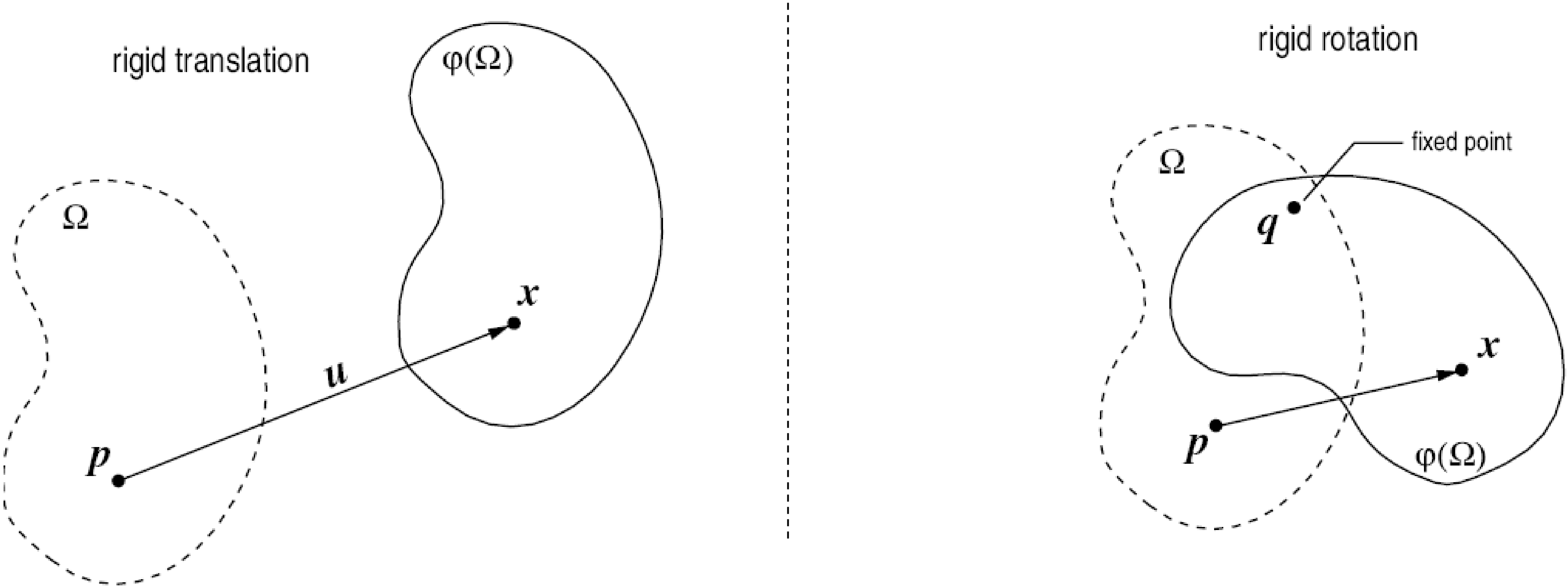


Figure 3.2. Rigid deformations.

Motion. Time-dependent deformations

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{p}, t) \qquad \dot{\boldsymbol{x}}(\boldsymbol{p}, t) = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{p}, t)}{\partial t}$$

$$\boldsymbol{\varphi}(\boldsymbol{p}, t) = \boldsymbol{p} + \boldsymbol{u}(\boldsymbol{p}, t)$$

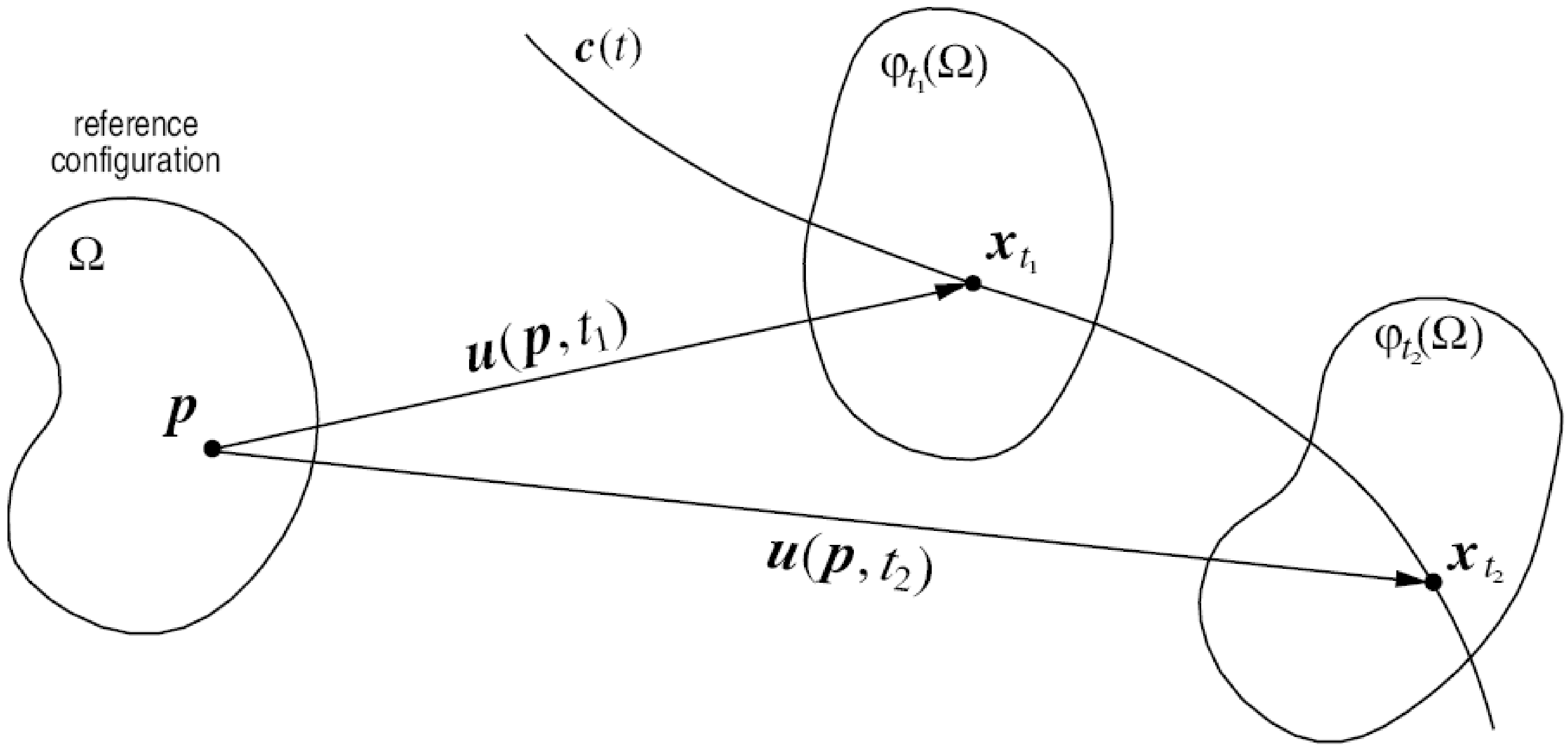


Figure 3.3. Motion.

Rigid motion. Rigid velocity

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{y}, t) + \boldsymbol{W}(t)\,(\boldsymbol{x} - \boldsymbol{y})$$

with $\boldsymbol{W}(t)$ a *skew* tensor

$$\Updownarrow$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{y}, t) + \boldsymbol{w}(t) \times (\boldsymbol{x} - \boldsymbol{y})$$

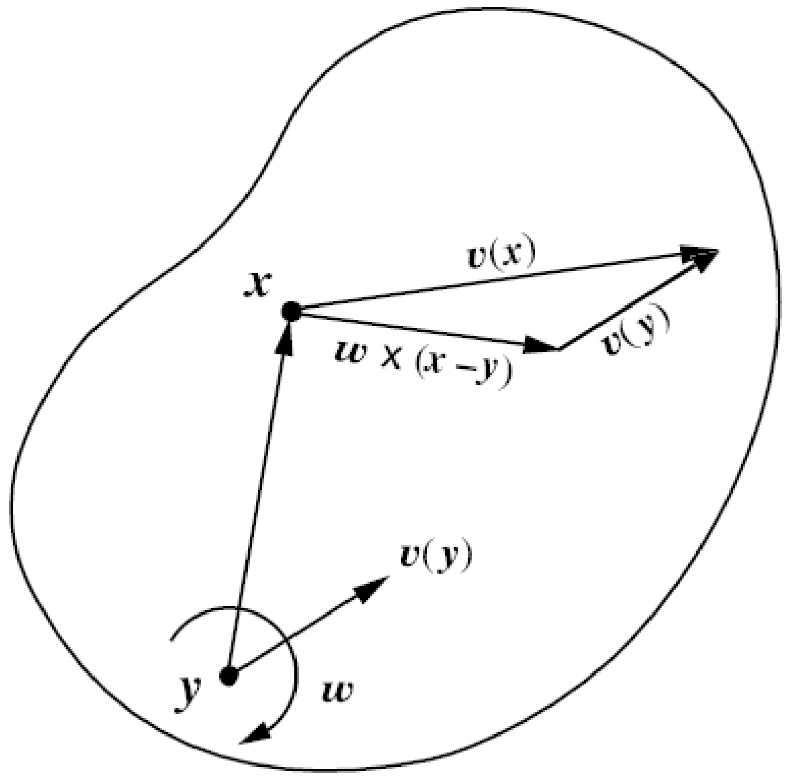


Figure 3.4. Rigid velocity.

The deformation gradient

$$\boldsymbol{F}(\boldsymbol{p},t) = \nabla_p \boldsymbol{\varphi}(\boldsymbol{p},t) = \frac{\partial \boldsymbol{x}_t}{\partial \boldsymbol{p}} \quad \Longleftrightarrow \quad \boldsymbol{F} = \boldsymbol{I} + \nabla_p \boldsymbol{u}$$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\ \mathrm{d}\boldsymbol{p}$$

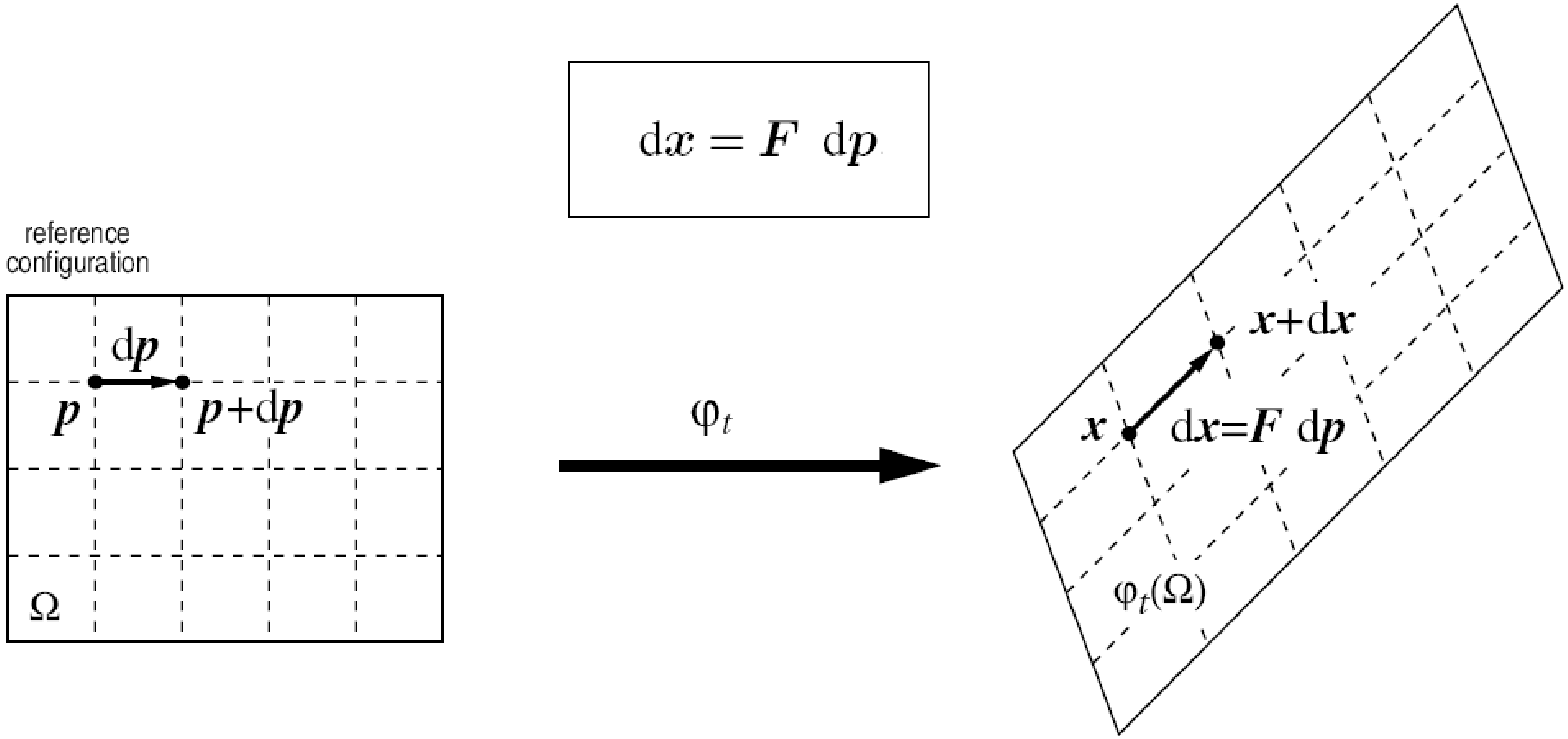


Figure 3.6. The deformation gradient.

Measuring volumetric changes. The determinant of the deformation gradient

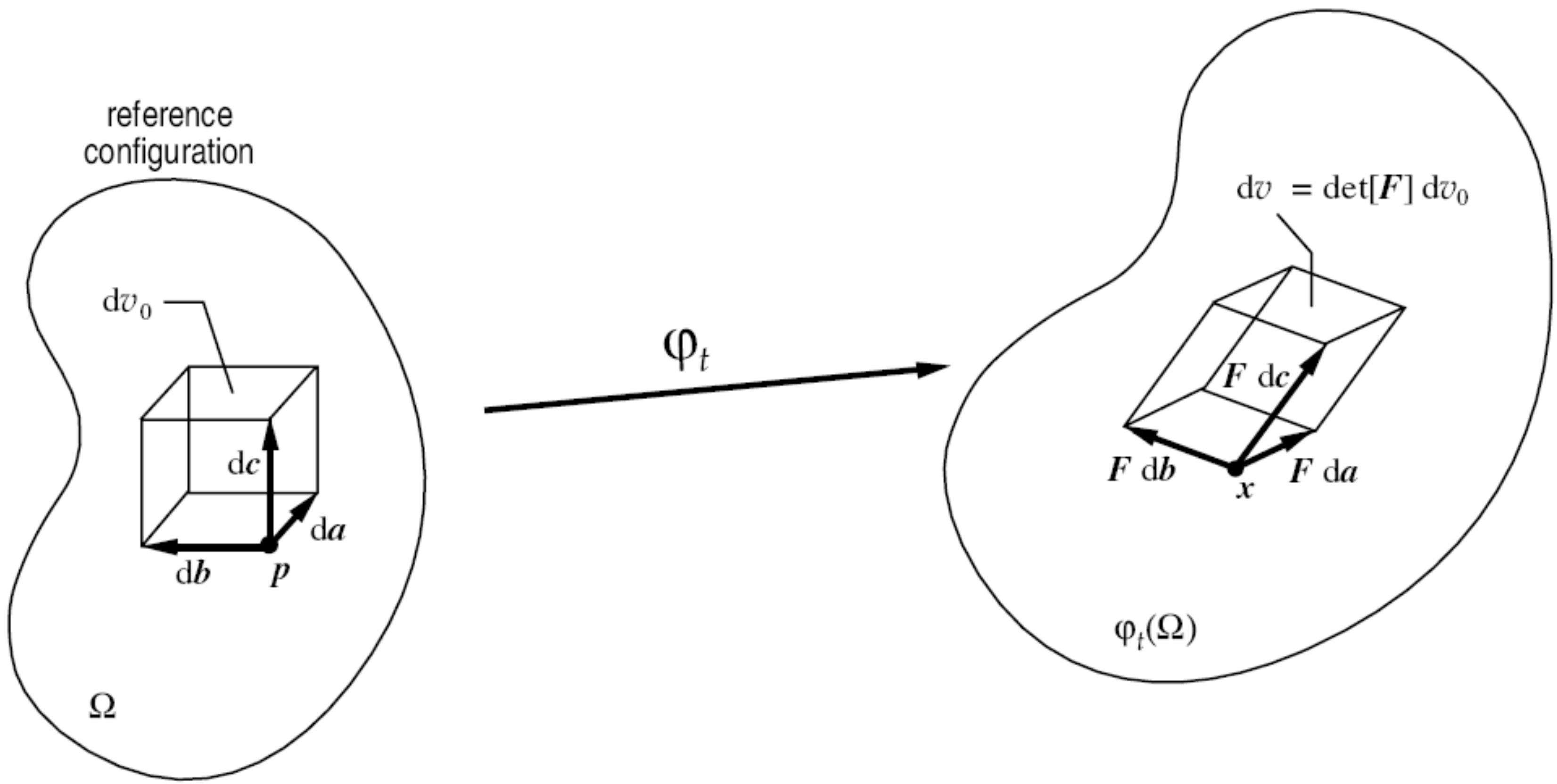
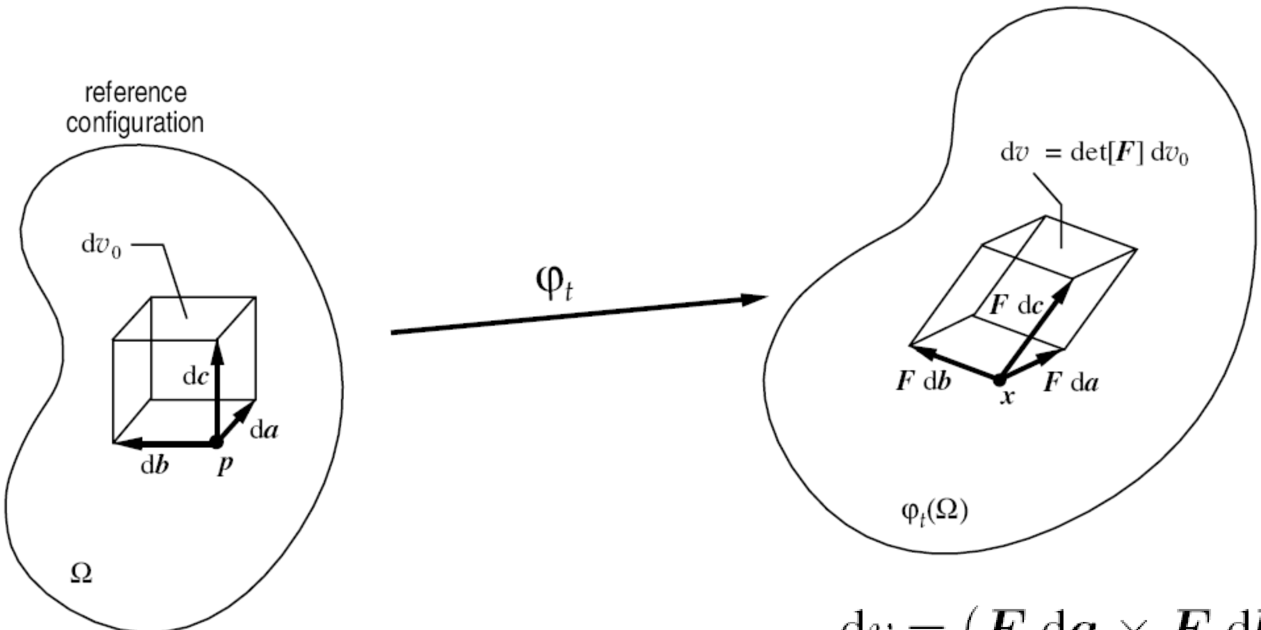


Figure 3.7. The determinant of the deformation gradient.



$$dv = (\boldsymbol{F}\, d\boldsymbol{a} \times \boldsymbol{F}\, d\boldsymbol{b}) \cdot \boldsymbol{F}\, d\boldsymbol{c}$$

$$dv_0 = (d\boldsymbol{a} \times d\boldsymbol{b}) \cdot d\boldsymbol{c}$$

$$\det \boldsymbol{T} = \frac{(\boldsymbol{T}\boldsymbol{u} \times \boldsymbol{T}\boldsymbol{v}) \cdot \boldsymbol{T}\boldsymbol{w}}{(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}} \quad \Rightarrow \quad \det \boldsymbol{F} = \frac{dv}{dv_0}$$

$$J \equiv \det \boldsymbol{F}$$

Locally **isochoric** (volume-preserving) deformations

$$J = 1$$

Locally **purely volumetric** deformations

$$\boldsymbol{F} = \alpha\, \boldsymbol{I}$$

$$\frac{l}{l_0} = \alpha \qquad \text{in } \textit{all directions.}$$

Isochoric/volumetric split of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{F}_{\text{iso}}\, \boldsymbol{F}_{\text{v}} = \boldsymbol{F}_{\text{v}}\, \boldsymbol{F}_{\text{iso}}$$

$$\boldsymbol{F}_{\text{v}} \equiv (\det \boldsymbol{F})^{\frac{1}{3}}\, \boldsymbol{I}$$

$$\boldsymbol{F}_{\text{iso}} \equiv (\det \boldsymbol{F})^{-\frac{1}{3}}\, \boldsymbol{F}$$

Note that

$$\det \boldsymbol{F}_{\text{v}} = [(\det \boldsymbol{F})^{\frac{1}{3}}]^3 \det \boldsymbol{I} = \det \boldsymbol{F}$$

$$\det \boldsymbol{F}_{\text{iso}} = [(\det \boldsymbol{F})^{-\frac{1}{3}}]^3 \det \boldsymbol{F} = 1$$

Polar decomposition. Local rotation and stretches

$$\boxed{\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}}$$

Right and left stretch tensors

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}}, \quad \boldsymbol{V} = \sqrt{\boldsymbol{B}},$$

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$$

Right and left Cauchy-Green strain tensors

$$\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}, \quad \boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^T$$

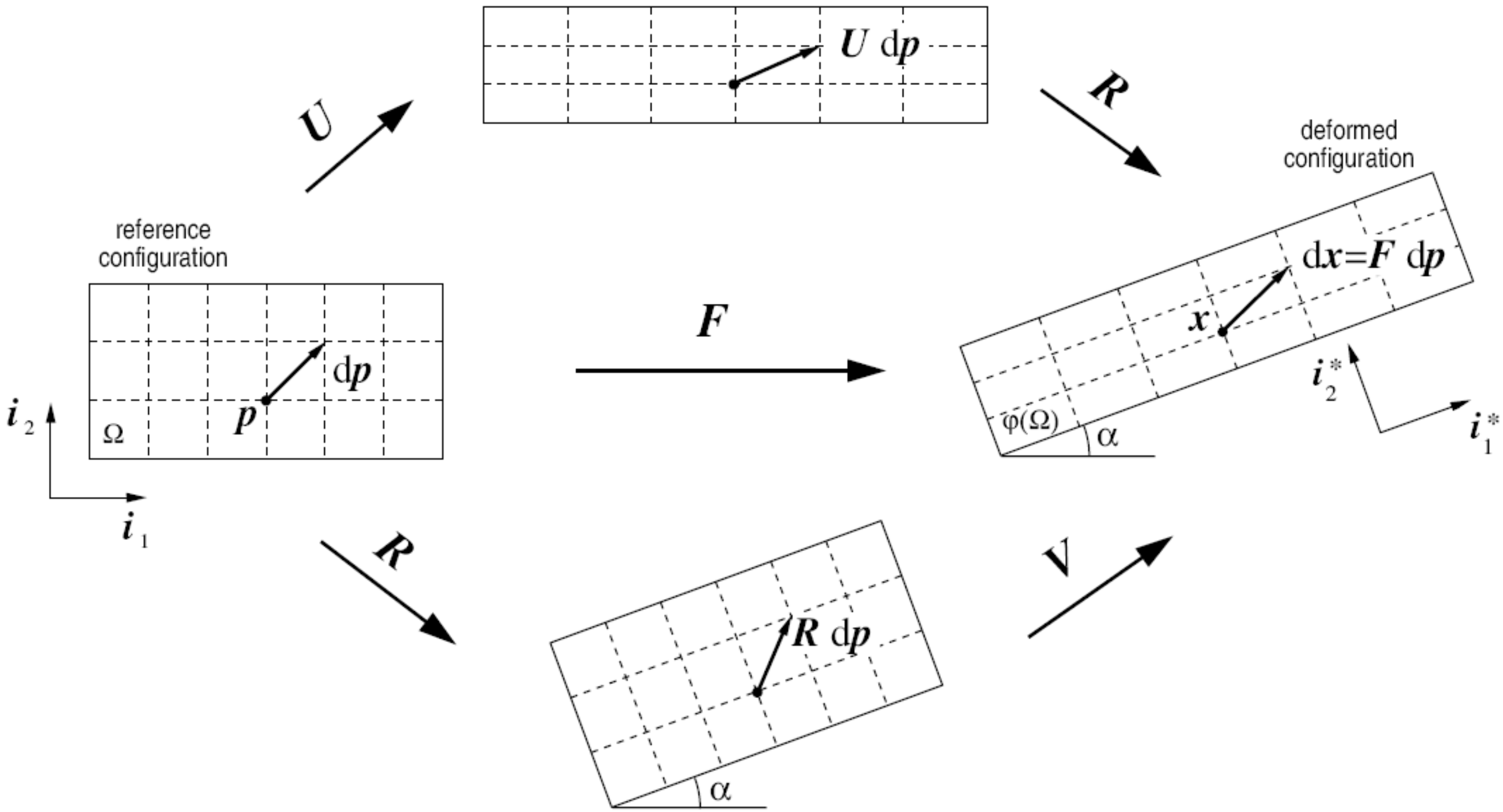


Figure 3.8. Polar decomposition of the deformation gradient. Stretches and rotation.

Spectral decomposition. Principal stretches

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{l}_i \otimes \boldsymbol{l}_i, \quad \boldsymbol{V} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

where the $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of $\boldsymbol{U}$ (and $\boldsymbol{V}$) named the *principal stretches*. The vectors $\boldsymbol{l}_i$ and $\boldsymbol{e}_i$ are unit eigenvectors of $\boldsymbol{U}$ and $\boldsymbol{V}$ respectively. The triads $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form orthonormal bases for the space $\mathscr{U}$ of vectors in $\mathscr{E}$. They are called, respectively, the *Lagrangian* and *Eulerian triads* and define the *Lagrangian* and *Eulerian principal directions*.

$$\boldsymbol{l}_i = \boldsymbol{R} \, \boldsymbol{e}_i$$

Strain measures

$$\|\mathrm{d}\boldsymbol{x}\|^2 = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p} \cdot \boldsymbol{F}\,\mathrm{d}\boldsymbol{p} = \boldsymbol{C}\,\mathrm{d}\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{p} = (\boldsymbol{I} + 2\,\boldsymbol{E}^{(2)})\,\mathrm{d}\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{p}$$

Green-Lagrange strain tensor

$$\begin{aligned} \boldsymbol{E}^{(2)} &= \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}\,) \\ &= \tfrac{1}{2}[\nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T + (\nabla_p \boldsymbol{u})^T \nabla_p \boldsymbol{u}] \end{aligned}$$

Locally rigid deformation

$$\|\mathrm{d}\boldsymbol{x}\| = \|\mathrm{d}\boldsymbol{p}\|, \quad \forall \ \mathrm{d}\boldsymbol{p} \iff \boldsymbol{C} = \boldsymbol{U} = \boldsymbol{I} \iff \boldsymbol{E}^{(2)} = \boldsymbol{0} \iff \boldsymbol{F} = \boldsymbol{R}$$

Spectral representation of the Green-Lagrange strain

$$\boldsymbol{E}^{(2)} = \sum_{i=1}^{3} \tfrac{1}{2}(\lambda_i^2 - 1)\, \boldsymbol{l}_i \otimes \boldsymbol{l}_i$$

Other Lagrangian strain tensors

$$\boldsymbol{E}^{(m)} = \begin{cases} \dfrac{1}{m}(\boldsymbol{U}^{\,m} - \boldsymbol{I}) & m \neq 0 \\ \ln[\boldsymbol{U}] & m = 0 \end{cases}$$

$$\boldsymbol{E}^{(m)} = \sum_{i=1}^{3} f(\lambda_i)\, \boldsymbol{l}_i \otimes \boldsymbol{l}_i,$$

$$f(\lambda_i) = \begin{cases} \dfrac{1}{m}(\lambda_i^m - 1) & m \neq 0 \\ \ln \lambda_i & m = 0 \end{cases}$$

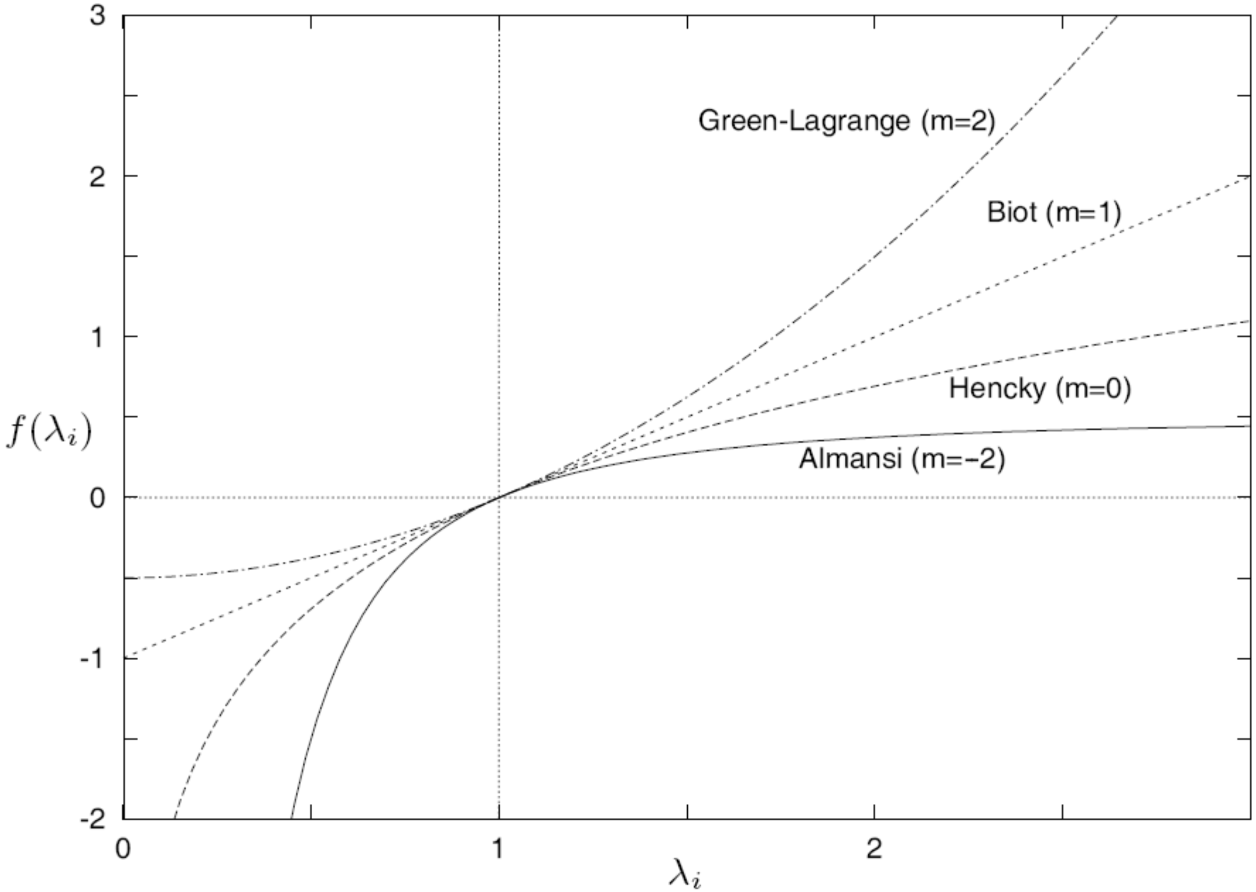


Figure 3.9. Strain measures. Principal strain as a function of the principal stretch for various strain measures.

Eulerian strain measures

$$\boldsymbol{\varepsilon}^{(m)} = \begin{cases} \dfrac{1}{m}(\boldsymbol{V}^m - \boldsymbol{I}\,) & m \neq 0 \\ \ln[\boldsymbol{V}] & m = 0 \end{cases}$$

$$\boldsymbol{\varepsilon}^{(m)} = \sum_{i=1}^{3} \; f(\lambda_i)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i\,.$$

$$\boldsymbol{\varepsilon}^{(m)} = \boldsymbol{R}\,\boldsymbol{E}^{(m)}\,\boldsymbol{R}^T$$

Velocity gradient

$$\boldsymbol{L} = \nabla_x \boldsymbol{v}$$

Rate of deformation and spin

Consider a uniform velocity gradient. We have

or, equivalently,

rigid velocity

$$\boldsymbol{v}^R(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{y}, t) + \boldsymbol{W}(t)\,(\boldsymbol{x} - \boldsymbol{y})$$

$$\boldsymbol{v}^S(\boldsymbol{x}, t) = \boldsymbol{D}(t)\,(\boldsymbol{x} - \boldsymbol{y}).$$

straining velocity

$$\boldsymbol{v}^S(\boldsymbol{x},t) = \boldsymbol{D}(t)\,(\boldsymbol{x}-\boldsymbol{y})$$

$$\boldsymbol{D} = \sum_{i=1}^{3} d_i\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

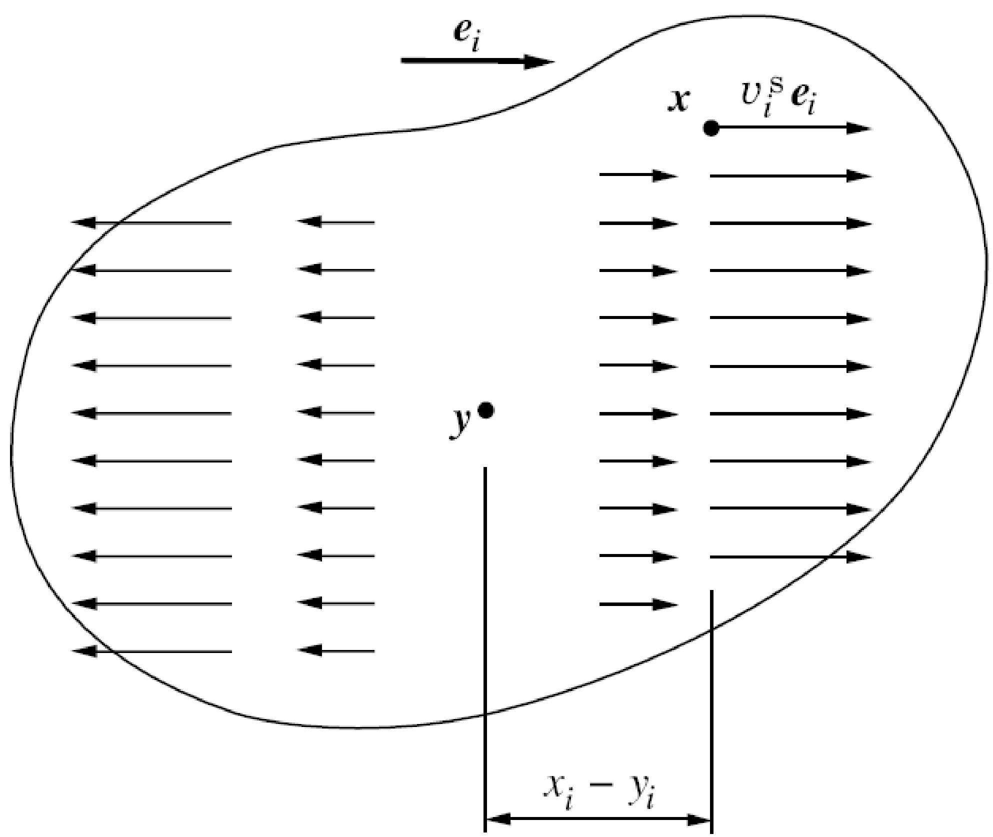


Figure 3.10. Straining velocity field.

Rate of volume change

$$\dot{J} \equiv (\det \boldsymbol{F})^{\dot{}} = \frac{\partial(\det \boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = J\, \boldsymbol{F}^{-T} : \dot{\boldsymbol{F}}$$

$$\mathrm{tr}\, \boldsymbol{L} = \mathrm{tr}\, \boldsymbol{D}$$

$$\boxed{\dot{J} = J\, \mathrm{tr}\, \boldsymbol{D}}$$

Also note that

$$\mathrm{tr}\, \boldsymbol{D} = \mathrm{div}_x\, \boldsymbol{v}$$

so that, equivalently,

$$\dot{J} = J\, \mathrm{div}_x\, \boldsymbol{v}$$

Stress Measures

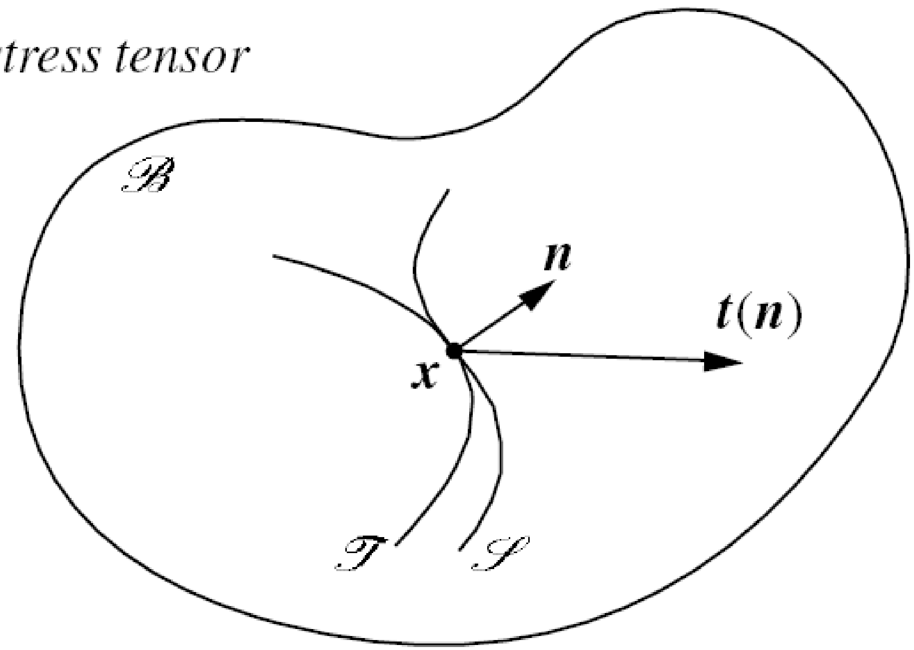
The Cauchy stress tensor

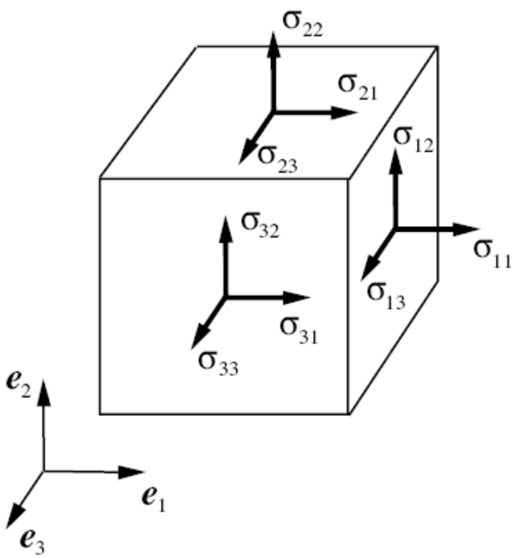
Cauchy's Theorem establishes that

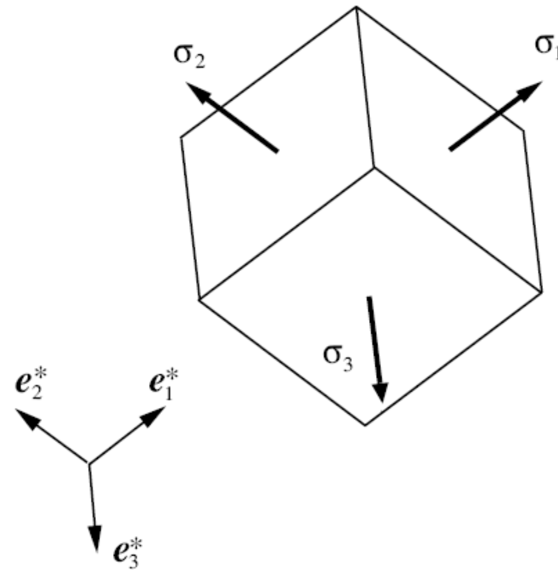
$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x})\, \boldsymbol{n}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

The tensor $\boldsymbol{\sigma}$ is called the *Cauchy stress tensor*







Principal Cauchy stresses

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \, \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$$

Deviatoric and hydrostatic Cauchy stresses

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\,\boldsymbol{I},$$

$$p \equiv \tfrac{1}{3}\, I_1(\boldsymbol{\sigma}) = \tfrac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma}$$

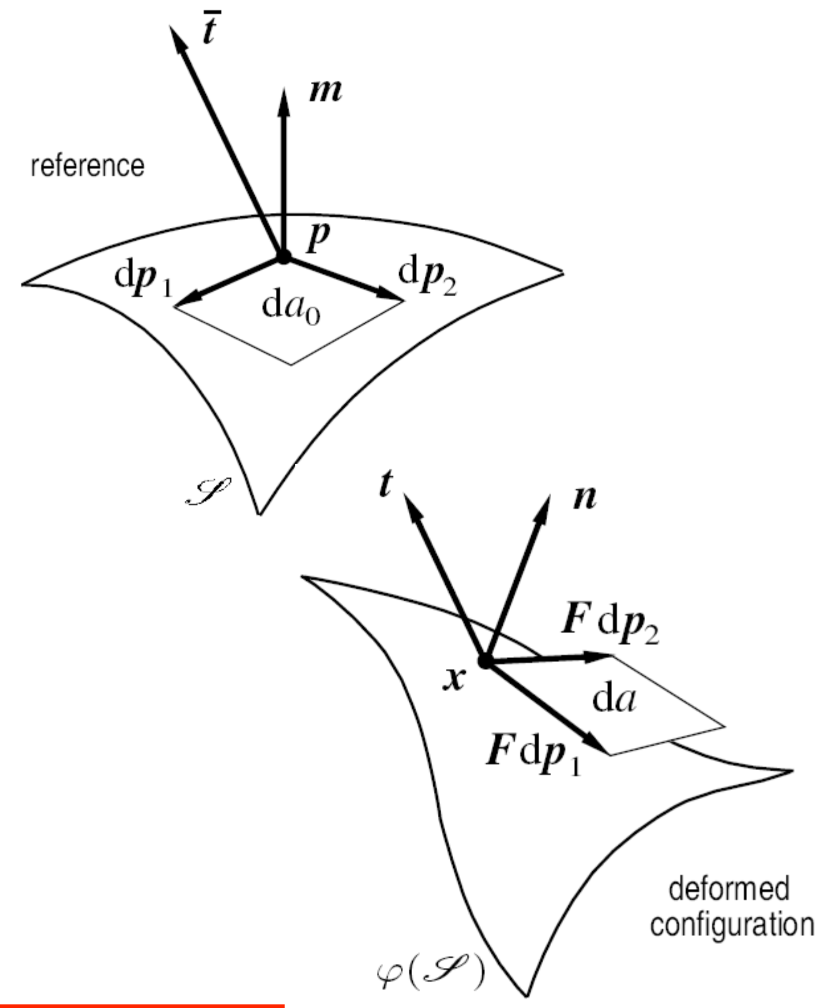
$$\boldsymbol{s} \equiv \boldsymbol{\sigma} - p\,\boldsymbol{I} = \mathsf{I}_{\mathrm{d}} : \boldsymbol{\sigma}$$

The Kirchhoff stress tensor

$$\boldsymbol{\tau} \equiv J\,\boldsymbol{\sigma}$$

The First Piola-Kirchhoff (or nominal) stress tensor

measures force per unit **reference** area



$$\bar{t} = \frac{da}{da_0}\, t = \frac{da}{da_0}\, \sigma\, n$$

$$m\, da_0 = dp_1 \times dp_2$$

$$n\, da = F\, dp_1 \times F\, dp_2$$

$$Su \times Sv = (\det S)\, S^{-T}\, (u \times v)$$

$$F^{T} n\, da = J\, dp_1 \times dp_2 = J\, m\, da_0$$

$$\frac{da}{da_0}\, n = J\, F^{-T} m$$

$$\bar{t} = J\, \sigma\, F^{-T}\, m \quad \Longrightarrow \quad \bar{t} = P\, m \qquad \boxed{P \equiv J\, \sigma\, F^{-T}}$$

Finite Strain Hyperelasticity

Dissipation inequality. Second law of thermodynamics

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho(\dot{\psi} + s\,\dot{\theta}) - \frac{1}{\theta}\,\boldsymbol{q} \cdot \boldsymbol{g} \geq 0$$

Thermodynamics with internal variables. Dissipative models

Free-energy function

$$\psi = \psi(\boldsymbol{F},\ \theta,\ \boldsymbol{\alpha})$$

Dissipation

$$\left(\boldsymbol{\sigma}\,\boldsymbol{F}^{-T} - \rho\,\frac{\partial\psi}{\partial\boldsymbol{F}}\right) : \dot{\boldsymbol{F}} - \rho\left(s + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} - \rho\,\frac{\partial\psi}{\partial\alpha_k}\,\dot{\alpha}_k - \frac{1}{\theta}\,\boldsymbol{q} \cdot \boldsymbol{g} \geq 0$$

$$\Updownarrow$$

$$\left(\boldsymbol{P} - \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{F}}\right) : \dot{\boldsymbol{F}} - \bar{\rho}\left(s + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} - \bar{\rho}\,\frac{\partial\psi}{\partial\alpha_k}\,\dot{\alpha}_k - \frac{J}{\theta}\,\boldsymbol{q} \cdot \boldsymbol{g} \geq 0$$

…the dissipation inequality implies

$$\boldsymbol{P} = \bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}, \qquad s = -\frac{\partial \psi}{\partial \theta}$$

$$\Updownarrow$$

$$\boldsymbol{\sigma} = \frac{1}{J}\,\bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}\,\boldsymbol{F}^{T} \quad \Longleftrightarrow \quad \boldsymbol{\tau}(\boldsymbol{F}) = \bar{\rho}\,\frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}}\,\boldsymbol{F}^{T}$$

Hyperelasticity. Definition

$$\boxed{\psi = \psi(\boldsymbol{F})} \quad \Longrightarrow \quad \text{No dissipation !}$$

Material objectivity

$$\psi(\boldsymbol{QF}) = \psi(\boldsymbol{F}) \quad \Longrightarrow \quad \psi(\boldsymbol{F}) = \psi(\boldsymbol{U})$$

$$\psi(\boldsymbol{F}) = \tilde{\psi}(\boldsymbol{C}) \equiv \psi(\sqrt{\boldsymbol{C}})$$

$$\boldsymbol{P} = \bar{\rho}\,\frac{\partial\tilde{\psi}}{\partial\boldsymbol{C}} : \frac{\partial\boldsymbol{C}}{\partial\boldsymbol{F}} = 2\bar{\rho}\boldsymbol{F}\,\frac{\partial\tilde{\psi}}{\partial\boldsymbol{C}}$$

$$\boldsymbol{\tau} = 2\bar{\rho}\boldsymbol{F}\,\frac{\partial\tilde{\psi}}{\partial\boldsymbol{C}}\,\boldsymbol{F}^{T}$$

$$\boldsymbol{\sigma} = \frac{2\bar{\rho}}{J}\,\boldsymbol{F}\,\frac{\partial\tilde{\psi}}{\partial\boldsymbol{C}}\,\boldsymbol{F}^{T}$$

Isotropic hyperelasticity

$$\psi(\boldsymbol{F}\,\boldsymbol{Q}) = \psi(\boldsymbol{F}) \quad \Longrightarrow \quad \psi(\boldsymbol{F}) = \psi(\boldsymbol{V})$$

$$\tilde{\psi}(\boldsymbol{B}) = \psi(\sqrt{\boldsymbol{B}})$$

$$\psi(\boldsymbol{F}) = \psi(\boldsymbol{U}\,) = \psi(\boldsymbol{V}) = \tilde{\psi}(\boldsymbol{C}) = \tilde{\psi}(\boldsymbol{B})$$

$$\Downarrow$$

$$\psi(\boldsymbol{U}\,) = \psi(\boldsymbol{R}\,\boldsymbol{U}\,\boldsymbol{R}^T\,)$$

the free-energy is an isotropic scalar function of a tensor argument

Principal stretches representation

$$\psi(\boldsymbol{V}) = \hat{\psi}(\lambda_1, \lambda_2, \lambda_3)$$

$$\tilde{\psi}(\boldsymbol{B}) = \check{\psi}(b_1, b_2, b_3) = \hat{\psi}(b_1^{\frac{1}{2}}, b_2^{\frac{1}{2}}, b_3^{\frac{1}{2}})$$

Principal stresses

$$\tau_i = 2\bar{\rho}\,\frac{\partial\check{\psi}}{\partial b_i}\,b_i = \bar{\rho}\,\frac{\partial\hat{\psi}}{\partial\lambda_i}\,\lambda_i$$

$$\boldsymbol{\tau} = \sum_i \bar{\rho}\,\frac{\partial\hat{\psi}}{\partial\lambda_i}\lambda_i\,\boldsymbol{e}_i \otimes \boldsymbol{e}_i$$

Invariant representation

$$\tilde{\psi}(\boldsymbol{B}) = \bar{\psi}(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))$$

$$\begin{aligned} I_1(\boldsymbol{S}) &\equiv \operatorname{tr} \boldsymbol{S} = S_{ii} \\ I_2(\boldsymbol{S}) &\equiv \tfrac{1}{2}[(\operatorname{tr} \boldsymbol{S})^2 - \operatorname{tr}(\boldsymbol{S}^2)] = \tfrac{1}{2}(S_{ii}S_{jj} - S_{ij}S_{ji}) \\ I_3(\boldsymbol{S}) &\equiv \det \boldsymbol{S} = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{pqr}S_{ip}S_{jq}S_{kr}. \end{aligned}$$

Stress

$$\boldsymbol{\tau} = J\,(\beta_0\,\boldsymbol{I} + \beta_1\,\boldsymbol{B} + \beta_{-1}\boldsymbol{B}^{-1})$$

$$\begin{aligned} \beta_0 &= \frac{2}{\sqrt{I_3}}\left[I_2\,\bar{\rho}\,\frac{\partial\bar{\psi}}{\partial I_2} + I_3\,\bar{\rho}\,\frac{\partial\bar{\psi}}{\partial I_3}\right] \\ \beta_1 &= \frac{2}{\sqrt{I_3}}\bar{\rho}\,\frac{\partial\bar{\psi}}{\partial I_1} \\ \beta_{-1} &= -2\sqrt{I_3}\,\bar{\rho}\,\frac{\partial\bar{\psi}}{\partial I_2}. \end{aligned}$$

Regularised (compressible) Neo-Hookean model

$$\bar{\rho}\,\bar{\psi}^*(I_1^*, J) = \tfrac{1}{2}G\,(I_1^* - 3) + \tfrac{1}{2}K\,(\ln J)^2$$

$$I_1^* \equiv \mathrm{tr}\,\boldsymbol{B}_{\mathrm{iso}}$$

$$\boldsymbol{B}_{\mathrm{iso}} \equiv \boldsymbol{F}_{\mathrm{iso}}\,\boldsymbol{F}_{\mathrm{iso}}^T = (\det \boldsymbol{F})^{-\frac{2}{3}}\,\boldsymbol{F}\,\boldsymbol{F}^T$$

$$\boxed{\boldsymbol{\tau} = G\,\mathrm{dev}[\boldsymbol{B}_{\mathrm{iso}}] + K\,(\ln J)\,\boldsymbol{I}}$$

Regularised (compressible) Ogden model

$$\bar{\rho}\,\hat{\psi}^*(\lambda_1^*,\lambda_2^*,\lambda_3^*,J)=\sum_{p=1}^{N}\frac{\mu_p}{\alpha_p}[(\lambda_1^*)^{\alpha_p}+(\lambda_2^*)^{\alpha_p}+(\lambda_3^*)^{\alpha_p}-3]+\tfrac{1}{2}K\,(\ln J)^2$$

$$\tau_i=\sum_{p=1}^{N}\mu_p\,J^{-\alpha_p/3}[\lambda_i^{\alpha_p}-\tfrac{1}{3}(\lambda_1^{\alpha_p}+\lambda_2^{\alpha_p}+\lambda_3^{\alpha_p})]+K\ln J$$

Hencky model (logarithmic strain-based)

$$\bar{\rho}\,\psi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : \mathsf{D} : \boldsymbol{\varepsilon}$$

$$\mathsf{D} \equiv 2G\,\mathsf{I}_S + (K - \tfrac{2}{3}G)\boldsymbol{I} \otimes \boldsymbol{I}$$

$$\boldsymbol{\varepsilon} \equiv \ln \boldsymbol{V} = \tfrac{1}{2}\ln \boldsymbol{B}$$

$$\boldsymbol{\tau} = \bar{\rho}\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \mathsf{D} : \boldsymbol{\varepsilon}$$

Important properties

$$\varepsilon_{\mathrm{v}} \equiv \mathrm{tr}\,\boldsymbol{\varepsilon} = 0 \Longleftrightarrow \det \boldsymbol{F} = 1$$

$$\boldsymbol{\varepsilon}_{\mathrm{d}} = \ln\ \boldsymbol{V}_{\mathrm{iso}}$$

Hyperelasticity boundary value problem

Reference (or material) description

$$G(\boldsymbol{u}, \boldsymbol{\eta}) = \int_{\Omega} [\boldsymbol{P} : \nabla_p \boldsymbol{\eta} - \bar{\boldsymbol{b}} \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\partial \Omega_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{\eta} \, \mathrm{d}a$$

Linearised virtual work equation

$$\mathrm{D}G(\boldsymbol{u}^*, \boldsymbol{\eta}) \, [\delta \boldsymbol{u}] = \int_{\Omega} \mathsf{A} : \nabla_p \delta \boldsymbol{u} : \nabla_p \boldsymbol{\eta} \, \mathrm{d}v$$

$$\mathsf{A} \equiv \left. \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}} \right|_{\boldsymbol{F}^*}$$

material tangent modulus or first elasticity tensor

$$\int_{\Omega} \mathsf{A} : \nabla_p \delta \boldsymbol{u} : \nabla_p \boldsymbol{\eta} \, \mathrm{d}v = -\int_{\Omega} (\boldsymbol{P} : \nabla_p \boldsymbol{\eta} - \bar{\boldsymbol{b}} \cdot \boldsymbol{\eta}) \, \mathrm{d}v + \int_{\partial \Omega_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{\eta} \, \mathrm{d}a$$

Spatial description

$$G(\boldsymbol{u},\boldsymbol{\eta}) = \int_{\varphi(\Omega)} [\boldsymbol{\sigma} : \nabla_x \boldsymbol{\eta} - \boldsymbol{b}\cdot\boldsymbol{\eta}]\,\mathrm{d}v - \int_{\varphi(\partial\Omega_t)} \boldsymbol{t}\cdot\boldsymbol{\eta}\,\mathrm{d}a$$

Linearised virtual work equation

$$\mathrm{D}G(\boldsymbol{u}^*,\boldsymbol{\eta})\,[\delta\boldsymbol{u}] = \int_{\varphi(\Omega)} \mathsf{a} : \nabla_x \delta\boldsymbol{u} : \nabla_x \boldsymbol{\eta}\,\mathrm{d}v$$

$$\mathsf{a}_{ijkl} = \frac{1}{J}\,\mathsf{A}_{imkn}\,F_{jm}\,F_{ln}$$

spatial tangent modulus

$$\mathsf{a}_{ijkl} = \frac{1}{J}\frac{\partial \tau_{ij}}{\partial F_{km}}\,F_{lm} - \sigma_{il}\,\delta_{jk}$$

$$\int_{\varphi(\Omega)} \mathsf{a} : \nabla_x \delta\boldsymbol{u} : \nabla_x \boldsymbol{\eta}\,\mathrm{d}v = -\int_{\varphi(\Omega)} [\boldsymbol{\sigma} : \nabla_x \boldsymbol{\eta} - \boldsymbol{b}\cdot\boldsymbol{\eta}]\,\mathrm{d}v + \int_{\varphi(\partial\Omega_t)} \boldsymbol{t}\cdot\boldsymbol{\eta}\,\mathrm{d}a$$

FE Equations (spatial description)

$$\mathbf{r}(\mathbf{u}_{n+1}) \equiv \mathbf{f}^{\,\text{int}}(\mathbf{u}_{n+1}) - \mathbf{f}^{\,\text{ext}}_{n+1} = \mathbf{0}$$

$$\mathbf{f}^{\,\text{int}}_{(e)} = \int_{\varphi_{n+1}(\Omega^{(e)})} \mathbf{B}^T\ \hat{\sigma}(\boldsymbol{\alpha}_n, \boldsymbol{F}(\mathbf{u}_{n+1}))\,\mathrm{d}v$$

$$\mathbf{f}^{\,\text{ext}}_{(e)} = \int_{\varphi_{n+1}(\Omega^{(e)})} \mathbf{N}^T\ \boldsymbol{b}_{n+1}\,\mathrm{d}v + \int_{\varphi_{n+1}(\partial\Omega^{(e)}_t)} \mathbf{N}^T\ \boldsymbol{t}_{n+1}\,\mathrm{d}a$$

Newton-Raphson solution

$$\mathbf{K}_T\ \delta\mathbf{u}^{(k)} = -\mathbf{r}^{(k-1)}$$

$$\mathbf{K}^{(e)}_T = \int_{\varphi^{(k)}_{n+1}(\Omega^{(e)})} \mathbf{G}^T\ \mathbf{a}\,\mathbf{G}\,\mathrm{d}v$$

Example

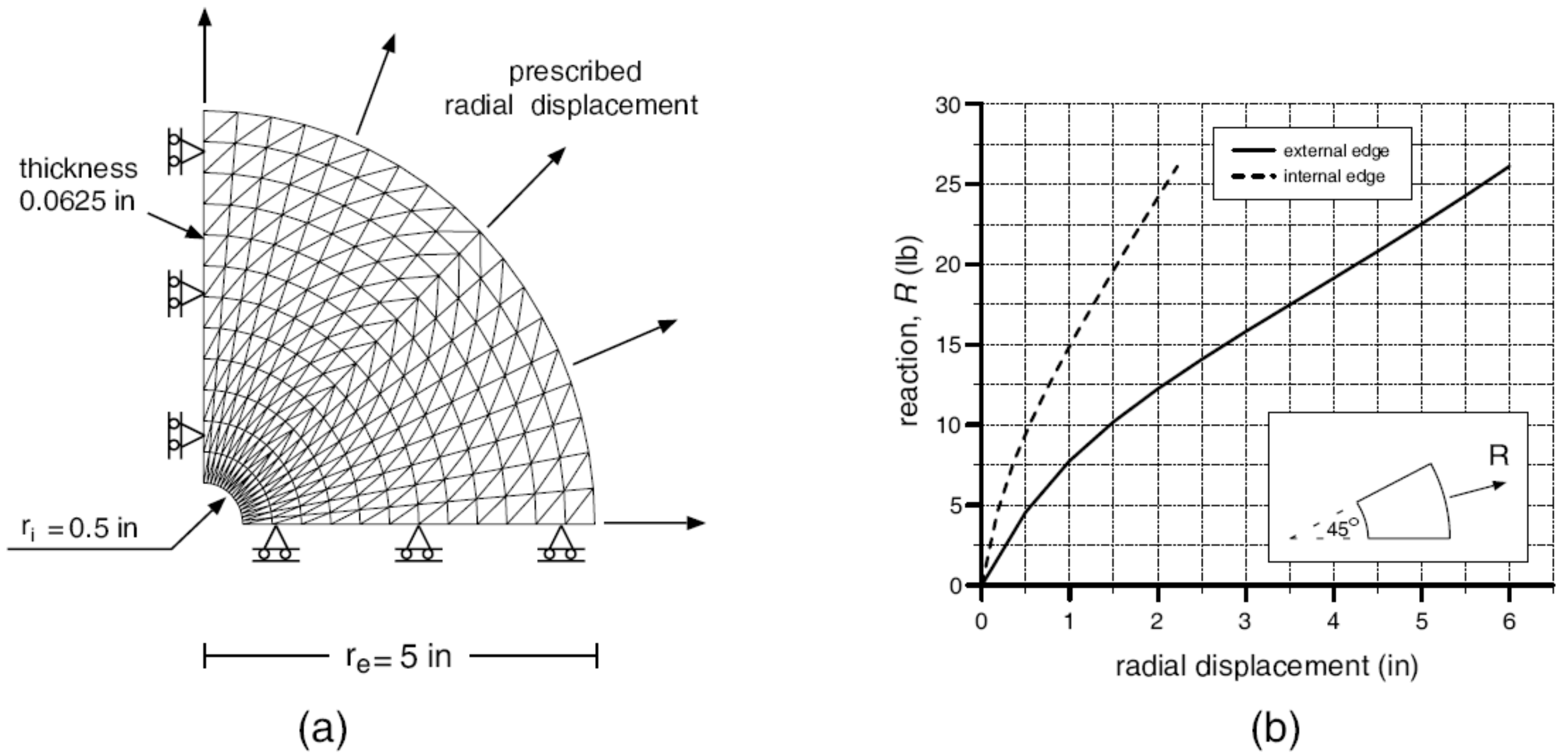


Figure 13.1. Annular plate: (a) geometry and boundary conditions; (b) reaction-displacement diagram. (Reproduced with permission from Finite elasticity in spatial description: Linearization aspects with membrane applications, EA de Souza Neto, D Perić and DRJ Owen, *International Journal for Numerical Methods in Engineering*, Vol 38 © 1995 John Wiley & Sons, Ltd.)