Computational Treatment of von Mises Plasticity

Topics:

- *Integration algorithm. Elastic predictor-plastic corrector*
- *Consistent tangent operator*
- *Elasto-plastic Incremental Boundary Value Problem solution*

Problem 7.1 (The elastoplastic constitutive initial value problem). *Given the initial values* $\boldsymbol{\varepsilon}^e(t_0)$ *and* $\boldsymbol{\alpha}(t_0)$ *and given the history of the strain tensor,* $\boldsymbol{\varepsilon}(t)$, $t \in [t_0, T]$, *find the functions* $\boldsymbol{\varepsilon}^e(t)$, $\boldsymbol{\alpha}(t)$ *and* $\dot{\gamma}(t)$ *for the elastic strain, hardening internal variables set and plastic multiplier that satisfy the reduced general elastoplastic constitutive equations*

$$
\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^e(t) &= \dot{\boldsymbol{\varepsilon}}(t) - \dot{\gamma}(t)\, \boldsymbol{N}(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) \\
\dot{\boldsymbol{\alpha}}(t) &= \dot{\gamma}(t)\, \boldsymbol{H}(\boldsymbol{\sigma}(t), \boldsymbol{A}(t))
\end{aligned}
\tag{7.6}
$$

$$
\dot{\gamma}(t) \geq 0, \quad \Phi(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) \leq 0, \quad \dot{\gamma}(t)\, \Phi(\boldsymbol{\sigma}(t), \boldsymbol{A}(t)) = 0 \tag{7.7}
$$

for each instant $t \in [t_0, T]$, *with*

$$
\boldsymbol{\sigma}(t) = \bar{\rho}\, \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}\bigg|_t, \quad \boldsymbol{A}(t) = \bar{\rho}\, \frac{\partial \psi}{\partial \alpha}\bigg|_t. \tag{7.8}
$$

Integration Algorithm

Problem 7.2 (The incremental elastoplastic constitutive problem). *Given the values* ε^e_n *and* $\boldsymbol{\alpha}_n$*, of the elastic strain and internal variables set at the beginning of the pseudo-time interval* $[t_n, t_{n+1}]$*, and given the prescribed incremental strain* $\Delta\varepsilon$ *for this interval, solve the following system of algebraic equations*

$$\begin{aligned} \varepsilon^e_{n+1} &= \varepsilon^e_n + \Delta\varepsilon - \Delta\gamma\, \boldsymbol{N}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\gamma\, \boldsymbol{H}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \end{aligned} \tag{7.10}$$

for the unknowns ε^e_{n+1}, $\boldsymbol{\alpha}_{n+1}$ *and* $\Delta\gamma$*, subjected to the constraints*

$$\Delta\gamma \geq 0, \quad \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \leq 0, \quad \Delta\gamma\, \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) = 0, \tag{7.11}$$

where

$$\boldsymbol{\sigma}_{n+1} = \bar{\rho}\, \frac{\partial \psi}{\partial \varepsilon^e}\bigg|_{n+1}, \quad \boldsymbol{A}_{n+1} = \bar{\rho}\, \frac{\partial \psi}{\partial \alpha}\bigg|_{n+1}. \tag{7.12}$$

General elastic predictor/plastic corrector scheme

Elastic predictor

$$\boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1} = \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}$$

$$\boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} = \boldsymbol{\alpha}_{n}.$$

$$\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{e}}\bigg|^{\mathrm{trial}}_{n+1}, \qquad \boldsymbol{A}^{\mathrm{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\bigg|^{\mathrm{trial}}_{n+1}$$

Admissibility check

IF $\quad \Phi^{\mathrm{trial}} \equiv \Phi(\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}, \boldsymbol{A}^{\mathrm{trial}}_{n+1}) \leq 0 \quad \Longrightarrow \quad (\cdot)_{n+1} := (\cdot)^{\mathrm{trial}}_{n+1}$

ELSE

Plastic corrector

$$\boldsymbol{\varepsilon}^{e}_{n+1} = \boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1} - \Delta\gamma\,\boldsymbol{N}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1})$$

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} + \Delta\gamma\,\boldsymbol{H}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1})$$

$$\Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) = 0,$$

Box 7.1. Fully implicit elastic predictor/return-mapping algorithm for numerical integration of general elastoplastic constitutive equations.

(i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n, evaluate the *elastic trial state*

$$\boldsymbol{\varepsilon}^{e\,\text{trial}}_{n+1} = \boldsymbol{\varepsilon}^e_n + \Delta\boldsymbol{\varepsilon}$$

$$\boldsymbol{\alpha}^{\text{trial}}_{n+1} = \boldsymbol{\alpha}_n$$

$$\boldsymbol{\sigma}^{\text{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}\bigg|^{\text{trial}}_{n+1}, \qquad \boldsymbol{A}^{\text{trial}}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\bigg|^{\text{trial}}_{n+1}$$

(ii) Check plastic admissibility

$$\text{IF } \Phi(\boldsymbol{\sigma}^{\text{trial}}_{n+1}, \boldsymbol{A}^{\text{trial}}_{n+1}) \leq 0$$

$$\text{THEN set} \quad (\cdot)_{n+1} = (\cdot)^{\text{trial}}_{n+1} \quad \text{and EXIT}$$

(iii) Return mapping. Solve the system

$$\left\{\begin{matrix} \boldsymbol{\varepsilon}^e_{n+1} - \boldsymbol{\varepsilon}^{e\,\text{trial}}_{n+1} + \Delta\gamma\,\boldsymbol{N}_{n+1} \\ \boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}^{\text{trial}}_{n+1} - \Delta\gamma\,\boldsymbol{H}_{n+1} \\ \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) \end{matrix}\right\} = \left\{\begin{matrix} \boldsymbol{0} \\ \boldsymbol{0} \\ 0 \end{matrix}\right\}$$

for $\boldsymbol{\varepsilon}^e_{n+1}$, $\boldsymbol{\alpha}_{n+1}$ and $\Delta\gamma$, with

$$\boldsymbol{\sigma}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}\bigg|_{n+1}, \qquad \boldsymbol{A}_{n+1} = \bar{\rho}\,\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\bigg|_{n+1}$$

(iv) EXIT

Geometric interpretation

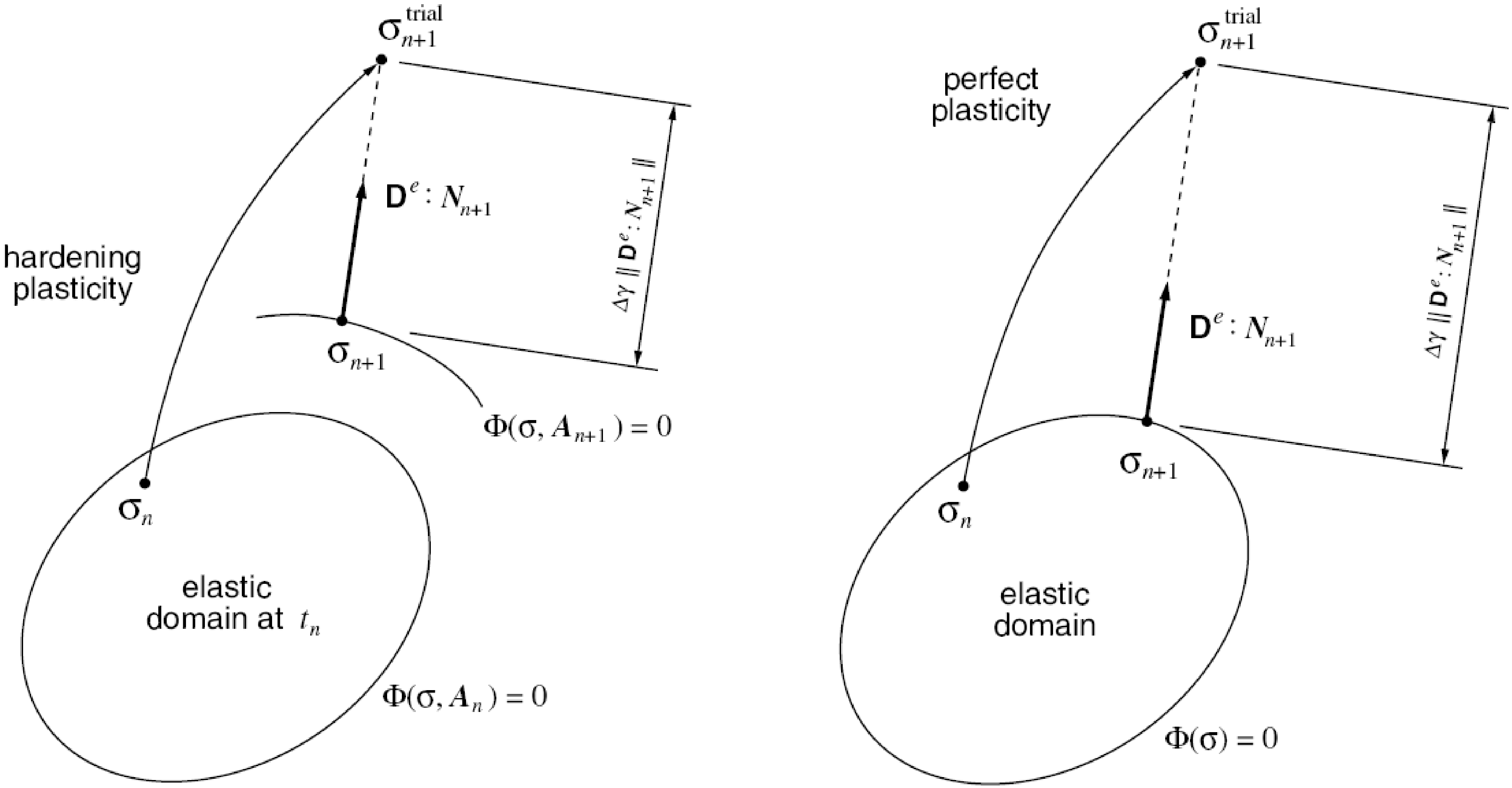

Figure 7.3. The fully implicit return mapping. Geometric interpretation for materials with linear elastic response.

Alternative procedures

Generalised Trapezoidal Return Mapping

$$\boxed{\begin{aligned} \varepsilon^e_{n+1} &= \varepsilon^{e\,\text{trial}}_{n+1} - \Delta\gamma\,[(1-\theta)\boldsymbol{N}_n + \theta\boldsymbol{N}_{n+1}] \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\gamma\,[(1-\theta)\boldsymbol{H}_n + \theta\boldsymbol{H}_{n+1}] \\ \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) &= 0, \end{aligned}}$$

$$0 \le \theta \le 1.$$

Generalised Mid-Point Return Mapping

$$\boxed{\begin{aligned} \varepsilon^e_{n+1} &= \varepsilon^{e\,\text{trial}}_{n+1} - \Delta\gamma\,\boldsymbol{N}_{n+\theta} \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\gamma\,\boldsymbol{H}_{n+\theta} \\ \Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) &= 0, \end{aligned}}$$

$$\begin{aligned} \boldsymbol{N}_{n+\theta} &= \boldsymbol{N}(\boldsymbol{\sigma}_{n+\theta}, \boldsymbol{A}_{n+\theta}) \\ \boldsymbol{H}_{n+\theta} &= \boldsymbol{H}(\boldsymbol{\sigma}_{n+\theta}, \boldsymbol{A}_{n+\theta}) \\ \boldsymbol{\sigma}_{n+\theta} &= (1-\theta)\boldsymbol{\sigma}_{n+1} + \theta\,\boldsymbol{\sigma}_n \\ \boldsymbol{A}_{n+\theta} &= (1-\theta)\boldsymbol{A}_n + \theta\,\boldsymbol{A}_{n+1} \end{aligned}$$

Application to the von Mises model

The model. von Mises with isotropic hardening

1. A linear elastic law

$$\boldsymbol{\sigma} = \mathsf{D}^e : \boldsymbol{\varepsilon}^e,$$

where D^e is the standard isotropic elasticity tensor.

2. A yield function of the form

$$\Phi(\boldsymbol{\sigma}, \sigma_y) = \sqrt{3\, J_2(\boldsymbol{s}(\boldsymbol{\sigma}))} - \sigma_y,$$

where

$$\sigma_y = \sigma_y(\bar{\varepsilon}^p)$$

is the uniaxial yield stress – a function of the accumulated plastic strain, $\bar{\varepsilon}^p$.

3. A standard associative flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\, \boldsymbol{N} = \dot{\gamma}\, \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}, \tag{7.76}$$

with the (Prandtl–Reuss) flow vector, $\boldsymbol{N}$, explicitly given by

$$\boldsymbol{N} \equiv \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \sqrt{\frac{3}{2}}\, \frac{\boldsymbol{s}}{\|\boldsymbol{s}\|}. \tag{7.77}$$

4. An associative hardening rule, with the evolution equation for the hardening internal variable given by

$$\dot{\bar{\varepsilon}}^p = \sqrt{\tfrac{2}{3}}\, \|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\gamma}. \tag{7.78}$$

… recall general scheme

Elastic predictor

$$\varepsilon^{e\ \mathrm{trial}}_{n+1} = \varepsilon^e_n + \Delta\varepsilon$$

$$\boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} = \boldsymbol{\alpha}_n .$$

$$\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1} = \bar{\rho}\, \frac{\partial\psi}{\partial\varepsilon^e}\bigg|^{\mathrm{trial}}_{n+1}, \qquad \boldsymbol{A}^{\mathrm{trial}}_{n+1} = \bar{\rho}\, \frac{\partial\psi}{\partial\alpha}\bigg|^{\mathrm{trial}}_{n+1}$$

Admissibility check

IF $\Phi^{\mathrm{trial}} \equiv \Phi(\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}, \boldsymbol{A}^{\mathrm{trial}}_{n+1}) \leq 0 \quad \Longrightarrow \quad (\cdot)_{n+1} := (\cdot)^{\mathrm{trial}}_{n+1}$

ELSE

Plastic corrector

$$\varepsilon^e_{n+1} = \varepsilon^{e\ \mathrm{trial}}_{n+1} - \Delta\gamma\, \boldsymbol{N}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1})$$

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} + \Delta\gamma\, \boldsymbol{H}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1})$$

$$\Phi(\boldsymbol{\sigma}_{n+1}, \boldsymbol{A}_{n+1}) = 0,$$

Elastic trial step

$$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n$$

$$\boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1} = \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}$$

$$\bar{\varepsilon}^{p\ \mathrm{trial}}_{n+1} = \bar{\varepsilon}^{p}_{n}. \quad \Longrightarrow \quad \sigma^{\mathrm{trial}}_{y\ n+1} = \sigma_y(\bar{\varepsilon}^{p}_{n}) = \sigma_{y\,n}$$

$$\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1} = \mathsf{D}^e : \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}$$

$$\boldsymbol{s}^{\mathrm{trial}}_{n+1} = 2G\ \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{\mathrm{d}\ n+1}, \qquad p^{\mathrm{trial}}_{n+1} = K\ \varepsilon^{e\ \mathrm{trial}}_{\mathrm{v}\ n+1}$$

Plastic admissibility check

IF $\Phi(\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}, \sigma_{y\,n}) \leq 0 \quad \Longrightarrow$

$$\boldsymbol{\varepsilon}^{e}_{n+1} = \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}$$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}$$

$$\bar{\varepsilon}^{p}_{n+1} = \bar{\varepsilon}^{p\ \mathrm{trial}}_{n+1} = \bar{\varepsilon}^{p}_{n}$$

$$\sigma_{y\ n+1} = \sigma^{\mathrm{trial}}_{y\ n+1} = \sigma_{y\,n}$$

ELSE… solve plastic corrector system of equations:

$$\left\{\begin{array}{l}
\varepsilon^e_{n+1} = \varepsilon^{e\ \mathrm{trial}}_{n+1} - \Delta\gamma \sqrt{\dfrac{3}{2}} \dfrac{s_{n+1}}{\|s_{n+1}\|} \\
\bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \Delta\gamma \\
\sqrt{3\, J_2(s_{n+1})} - \sigma_y(\bar{\varepsilon}^p_{n+1}) = 0,
\end{array}\right.$$

for ε^e_{n+1}, $\bar{\varepsilon}^p_{n+1}$ and $\Delta\gamma$ and where

$$s_{n+1} = s_{n+1}(\varepsilon^e_{n+1}) = 2G\,\mathrm{dev}[\varepsilon^e_{n+1}]$$

8 coupled scalar equations in 3-D
6 eqs in axisymmetric case
5 eqs in plane strain case

Single-equation return mapping

Volumetric/deviatoric split

$$\varepsilon^{e}_{v\;n+1} = \varepsilon^{e\;\text{trial}}_{v\;n+1}$$

$$\varepsilon^{e}_{d\;n+1} = \varepsilon^{e\;\text{trial}}_{d\;n+1} - \Delta\gamma\sqrt{\frac{3}{2}}\,\frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|}$$

$\Leftrightarrow$

$$p_{n+1} = p^{\text{trial}}_{n+1}$$

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{\text{trial}}_{n+1} - \Delta\gamma\; 2G\sqrt{\frac{3}{2}}\,\frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|}$$

$$\left(1 + \sqrt{\frac{3}{2}}\,\frac{\Delta\gamma\; 2G}{\|\boldsymbol{s}_{n+1}\|}\right)\boldsymbol{s}_{n+1} = \boldsymbol{s}^{\text{trial}}_{n+1}$$

$\Rightarrow$

$$\frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|} = \frac{\boldsymbol{s}^{\text{trial}}_{n+1}}{\|\boldsymbol{s}^{\text{trial}}_{n+1}\|}$$

$$\frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|} = \frac{\boldsymbol{s}_{n+1}^{\text{trial}}}{\|\boldsymbol{s}_{n+1}^{\text{trial}}\|} \qquad \left(1 + \sqrt{\frac{3}{2}}\frac{\Delta\gamma\, 2G}{\|\boldsymbol{s}_{n+1}\|}\right)\boldsymbol{s}_{n+1} = \boldsymbol{s}_{n+1}^{\text{trial}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \boldsymbol{s}_{n+1} &= \left(1 - \sqrt{\frac{3}{2}}\frac{\Delta\gamma\, 2G}{\|\boldsymbol{s}_{n+1}^{\text{trial}}\|}\right)\boldsymbol{s}_{n+1}^{\text{trial}} \\ &= \left(1 - \frac{\Delta\gamma\, 3G}{q_{n+1}^{\text{trial}}}\right)\boldsymbol{s}_{n+1}^{\text{trial}}, \end{aligned}}$$

… recall plastic consistency equation

$$\sqrt{3\,J_2(\boldsymbol{s}_{n+1})} - \sigma_y(\bar{\varepsilon}^p_{n+1}) = 0$$

combined with

$$\begin{aligned}\boldsymbol{s}_{n+1} &= \left(1 - \sqrt{\frac{3}{2}}\,\frac{\Delta\gamma\,2G}{\|\boldsymbol{s}^{\text{trial}}_{n+1}\|}\right)\boldsymbol{s}^{\text{trial}}_{n+1}\\ &= \left(1 - \frac{\Delta\gamma\,3G}{q^{\text{trial}}_{n+1}}\right)\boldsymbol{s}^{\text{trial}}_{n+1},\end{aligned}$$

gives the **single-equation** return mapping

$$\tilde{\Phi}(\Delta\gamma) \equiv q^{\text{trial}}_{n+1} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}^p_n + \Delta\gamma) = 0$$

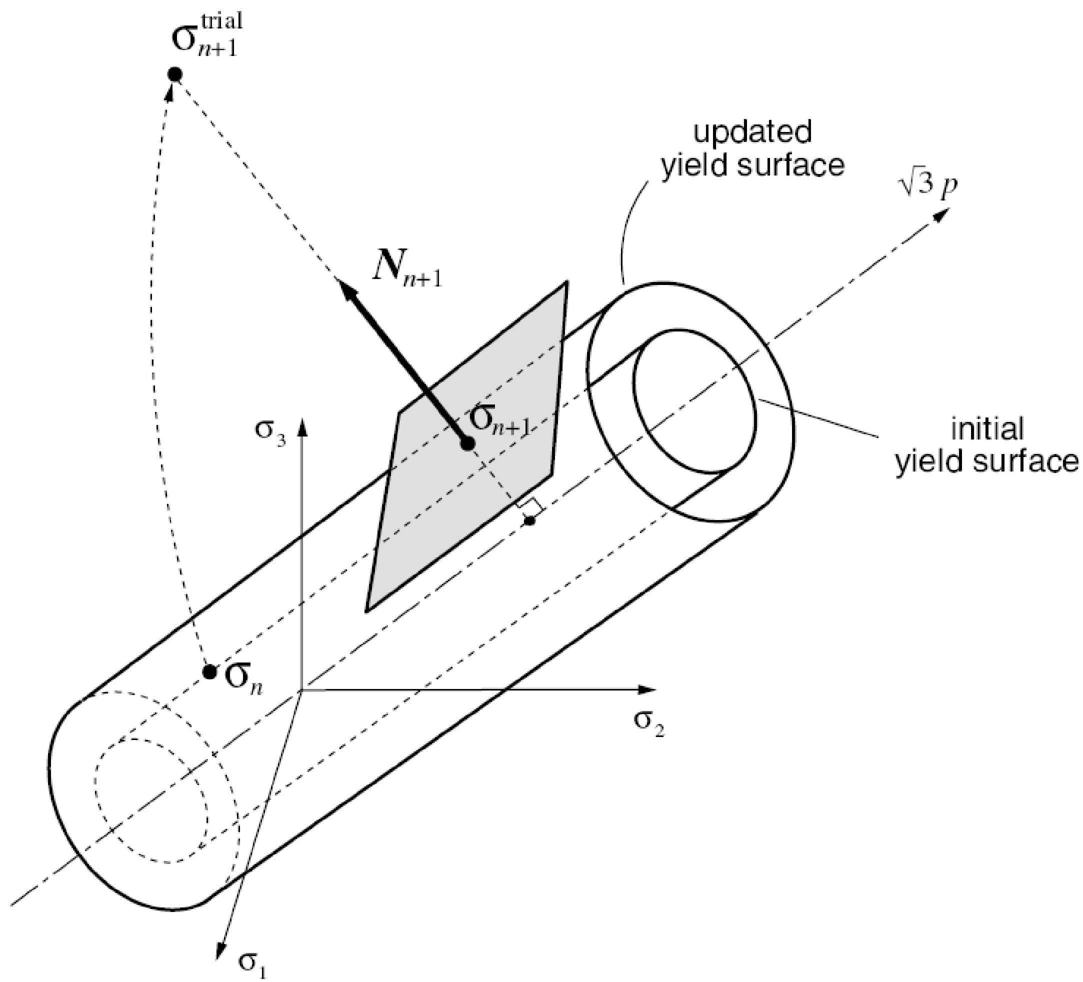

Figure 7.8. The implicit elastic predictor/return-mapping scheme for the von Mises model. Geometric interpretation in principal stress space.

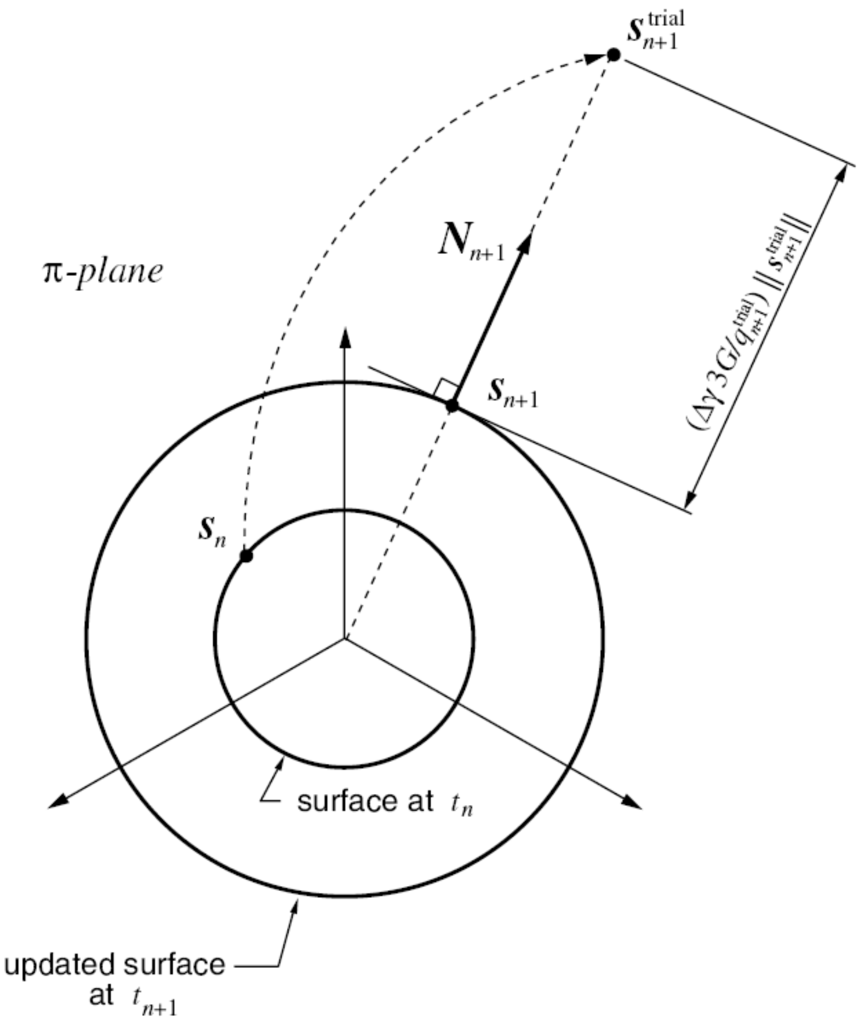

Figure 7.9. The implicit elastic predictor/return-mapping scheme for the von Mises model. Geometric interpretation in the deviatoric plane.

(i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n, evaluate the *elastic trial state*

$$\boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{n+1} := \boldsymbol{\varepsilon}^e_n + \Delta\boldsymbol{\varepsilon}$$

$$\bar{\varepsilon}^{p\,\mathrm{trial}}_{n+1} := \bar{\varepsilon}^p_n$$

$$p^{\mathrm{trial}}_{n+1} := K\,\varepsilon^{e\,\mathrm{trial}}_{\mathrm{v}\;n+1}; \qquad \boldsymbol{s}^{\mathrm{trial}}_{n+1} := 2G\,\boldsymbol{\varepsilon}^{e\,\mathrm{trial}}_{\mathrm{d}\;n+1}$$

$$q^{\mathrm{trial}}_{n+1} := \sqrt{\tfrac{3}{2}\,\boldsymbol{s}^{\mathrm{trial}}_{n+1} : \boldsymbol{s}^{\mathrm{trial}}_{n+1}}$$

(ii) Check plastic admissibility

IF $q^{\mathrm{trial}}_{n+1} - \sigma_y(\bar{\varepsilon}^{p\,\mathrm{trial}}_{n+1}) \leq 0$

THEN set $(\cdot)_{n+1} := (\cdot)^{\mathrm{trial}}_{n+1}$ and EXIT

(iii) Return mapping. Solve the equation

$$\tilde{\Phi}(\Delta\gamma) \equiv q^{\mathrm{trial}}_{n+1} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}^p_n + \Delta\gamma) = 0$$

for $\Delta\gamma$ using the Newton–Raphson method – GOTO Box 7.4 – and update the state variables

$$p_{n+1} := p^{\mathrm{trial}}_{n+1}; \qquad \boldsymbol{s}_{n+1} := \left(1 - \frac{\Delta\gamma\,3G}{q^{\mathrm{trial}}_{n+1}}\right)\boldsymbol{s}^{\mathrm{trial}}_{n+1}$$

$$\boldsymbol{\sigma}_{n+1} := \boldsymbol{s}_{n+1} + p_{n+1}\,\boldsymbol{I}$$

$$\boldsymbol{\varepsilon}^e_{n+1} = \frac{1}{2G}\,\boldsymbol{s}_{n+1} + \frac{1}{3}\,\varepsilon^{e\,\mathrm{trial}}_{\mathrm{v}\;n+1}\,\boldsymbol{I}$$

$$\bar{\varepsilon}^p_{n+1} := \bar{\varepsilon}^p_n + \Delta\gamma$$

(iv) EXIT

Algorithmic incremental constitutive function for the stress tensor

$$\boldsymbol{\sigma}_{n+1} = \bar{\boldsymbol{\sigma}}_{n+1}(\bar{\varepsilon}_n^p, \boldsymbol{\varepsilon}_{n+1}^{e\ \mathrm{trial}}) \equiv \left[\mathsf{D}^e - \hat{H}(\Phi^{\mathrm{trial}})\,\frac{\Delta\gamma\ 6G^2}{q_{n+1}^{\mathrm{trial}}}\,\mathsf{I}_{\mathrm{d}}\right] : \boldsymbol{\varepsilon}_{n+1}^{e\ \mathrm{trial}},$$

where $\hat{H}$ is the *Heaviside step function* defined as

$$\hat{H}(a) \equiv \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a \leq 0 \end{cases}, \qquad \text{for any scalar } a,$$

I_d is the deviatoric projection tensor defined by (3.94) (page 59),

$$\begin{aligned} q_{n+1}^{\mathrm{trial}} &= \sqrt{\tfrac{3}{2}}\,\|\boldsymbol{s}_{n+1}^{\mathrm{trial}}\| = 2G\sqrt{\tfrac{3}{2}}\,\|\boldsymbol{\varepsilon}_{\mathrm{d}\ n+1}^{e\ \mathrm{trial}}\| \\ &= q_{n+1}^{\mathrm{trial}}(\boldsymbol{\varepsilon}_{n+1}^{e\ \mathrm{trial}}) \equiv 2G\sqrt{\tfrac{3}{2}}\ \|\ \mathsf{I}_{\mathrm{d}} : \boldsymbol{\varepsilon}_{n+1}^{e\ \mathrm{trial}}\|, \end{aligned}$$

Return-mapping solution. Newton-Raphson Method

(i) Initialise iteration counter, $k := 0$, set initial guess for $\Delta\gamma$

$$\Delta\gamma^{(0)} := 0$$

and corresponding residual (yield function value)

$$\tilde{\Phi} := q_{n+1}^{\mathrm{trial}} - \sigma_y(\bar{\varepsilon}_n^p)$$

(ii) Perform Newton–Raphson iteration

$$H := \left.\frac{\mathrm{d}\sigma_y}{\mathrm{d}\bar{\varepsilon}^p}\right|_{\bar{\varepsilon}_n^p + \Delta\gamma} \qquad \text{(hardening slope)}$$

$$d := \frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}\Delta\gamma} = -3G - H \qquad \text{(residual derivative)}$$

$$\Delta\gamma := \Delta\gamma - \frac{\tilde{\Phi}}{d} \qquad \text{(new guess for } \Delta\gamma\text{)}$$

(iii) Check for convergence

$$\tilde{\Phi} := q_{n+1}^{\mathrm{trial}} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}_n^p + \Delta\gamma)$$

IF $|\tilde{\Phi}| \le \epsilon_{\mathrm{tol}}$ THEN RETURN to Box 7.3

(iv) GOTO (ii)

Finite step accuracy. Iso-error maps

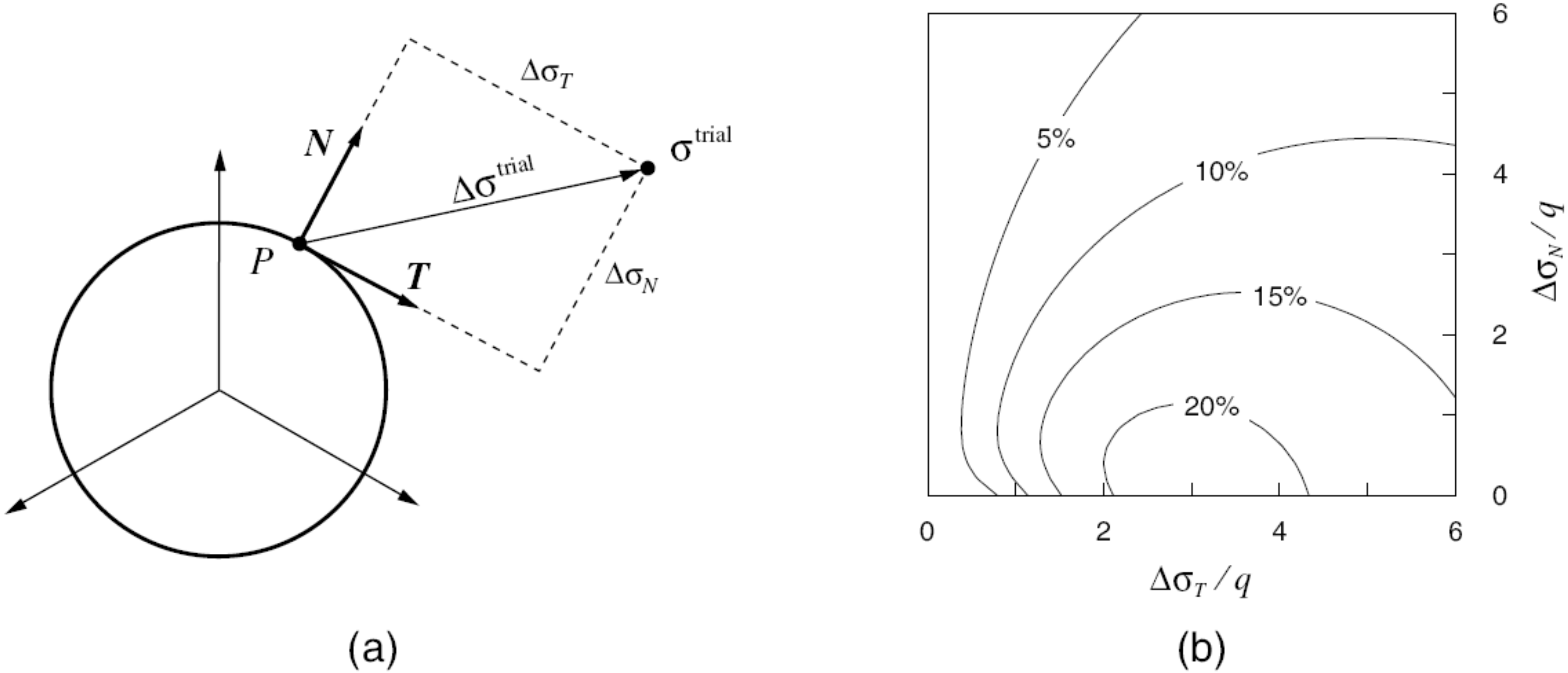

Figure 7.7. Iso-error map: (a) typical increment directions; and (b) a typical iso-error map.

Consistent Elasto-plastic tangent operator

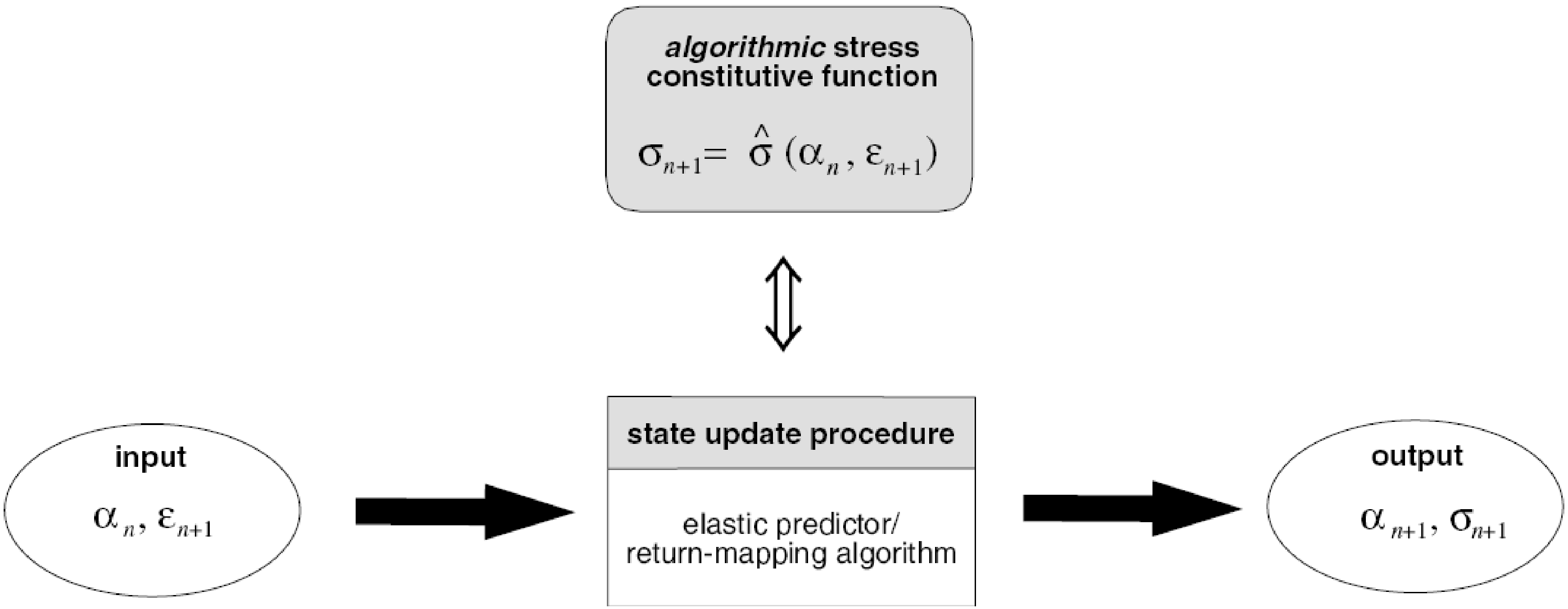

Figure 7.12. The algorithmic constitutive function for the stress tensor.

Algorithmic constitutive function for von Mises model (implicit algorithm)

$$\boldsymbol{\sigma}_{n+1} = \bar{\boldsymbol{\sigma}}_{n+1}(\bar{\varepsilon}^p_n, \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}) \equiv \left[\mathsf{D}^e - \hat{H}(\Phi^{\mathrm{trial}})\,\frac{\Delta\gamma\, 6G^2}{q^{\mathrm{trial}}_{n+1}}\,\mathsf{I}_{\mathrm{d}}\right] : \boldsymbol{\varepsilon}^{e\ \mathrm{trial}}_{n+1}$$

$$\Downarrow$$

$$\boldsymbol{\sigma}_{n+1} = \hat{\boldsymbol{\sigma}}_{n+1}(\bar{\varepsilon}^p_n, \boldsymbol{\varepsilon}^p_n, \boldsymbol{\varepsilon}_{n+1}) \equiv \bar{\boldsymbol{\sigma}}_{n+1}(\bar{\varepsilon}^p_n, \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_n)$$

Algorithmic constitutive function derivative

$$\boxed{\mathsf{D} \equiv \frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} = \left.\frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right|_{\alpha_n} \quad \Longleftarrow \quad \textbf{consistent tangent operator}}$$

…equivalently, we have

$$\mathsf{D} = \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \varepsilon_{n+1}} = \frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \varepsilon^{e\ \mathrm{trial}}_{n+1}}$$

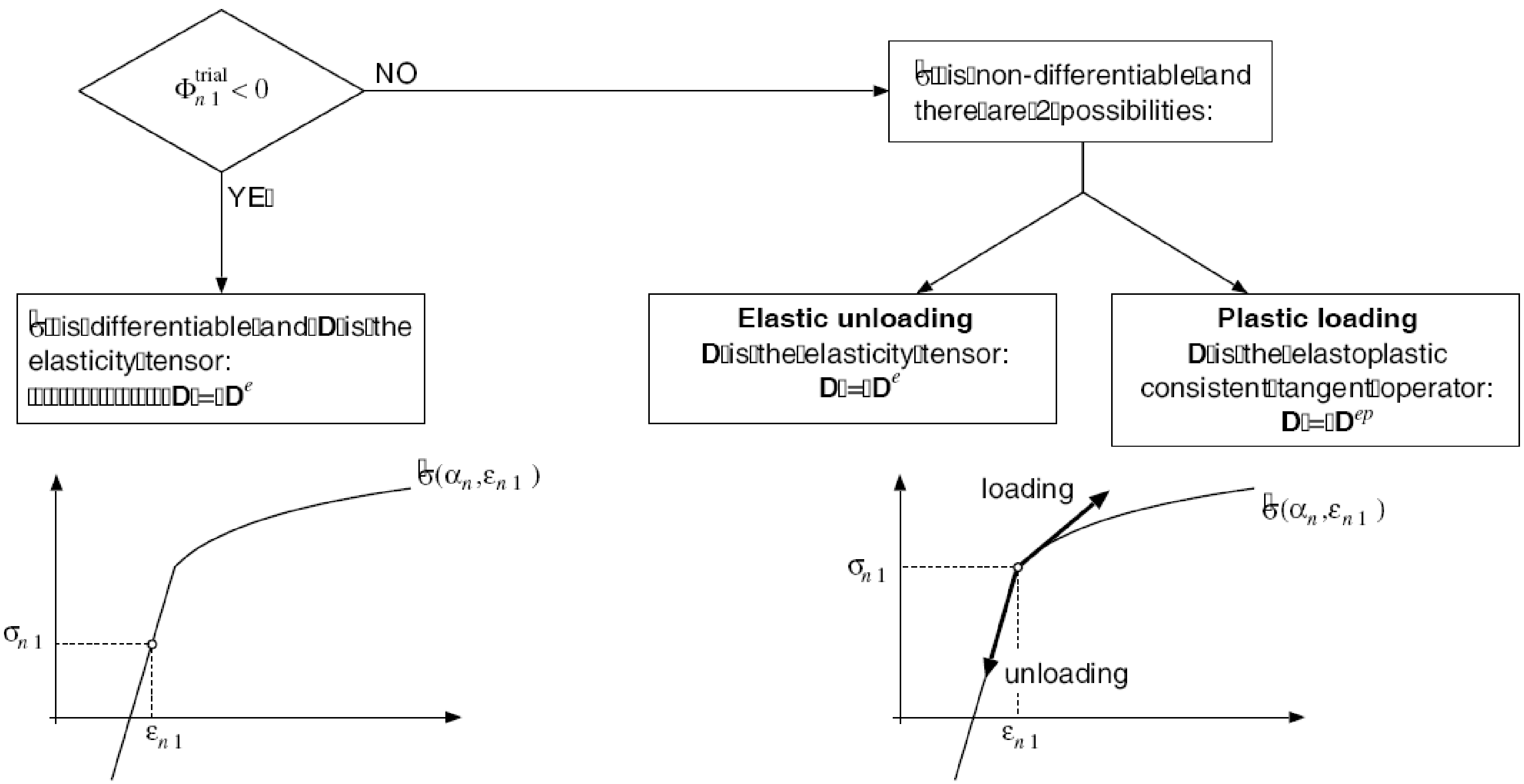

Figure 7.13. The tangent moduli consistent with elastic predictor/return-mapping integration algorithms.

Elasto-plastic consistent tangent operator. Derivation

$$\boldsymbol{\sigma}_{n+1} = \left[\mathsf{D}^e - \frac{\Delta\gamma\, 6G^2}{q_{n+1}^{\text{trial}}} \, \mathsf{I}_{\text{d}} \right] : \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}, \tag{7.113}$$

where $\Delta\gamma$ is the solution of the return-mapping equation of the algorithm (Box 7.3),

$$\tilde{\Phi}(\Delta\gamma) \equiv q_{n+1}^{\text{trial}} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}_n^p + \Delta\gamma) = 0. \tag{7.114}$$

A straightforward application of tensor differentiation rules to (7.113) gives

$$\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}} = \mathsf{D}^e - \frac{\Delta\gamma\, 6G^2}{q_{n+1}^{\text{trial}}} \, \mathsf{I}_{\text{d}} - \frac{6G^2}{q_{n+1}^{\text{trial}}} \, \boldsymbol{\varepsilon}_{\text{d}\ n+1}^{e\ \text{trial}} \otimes \frac{\partial \Delta\gamma}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}}$$
$$+ \frac{\Delta\gamma\, 6G^2}{(q_{n+1}^{\text{trial}})^2} \, \boldsymbol{\varepsilon}_{\text{d}\ n+1}^{e\ \text{trial}} \otimes \frac{\partial q_{n+1}^{\text{trial}}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}}.$$

…recall that

$$q_{n+1}^{\text{trial}} = \sqrt{\tfrac{3}{2}}\|\boldsymbol{s}_{n+1}^{\text{trial}}\| = 2G\sqrt{\tfrac{3}{2}}\|\boldsymbol{\varepsilon}_{\mathrm{d}\ n+1}^{e\ \text{trial}}\|$$

$$= q_{n+1}^{\text{trial}}(\boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}) \equiv 2G\sqrt{\tfrac{3}{2}}\ \|\ \mathbf{I}_{\mathrm{d}} : \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}\|$$

so that we obtain

$$\frac{\partial q_{n+1}^{\text{trial}}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}} = 2G\sqrt{\tfrac{3}{2}}\ \bar{\boldsymbol{N}}_{\ n+1}$$

where we have conveniently defined the *unit* flow vector

$$\bar{\boldsymbol{N}}_{n+1} \equiv \sqrt{\tfrac{2}{3}}\ \boldsymbol{N}_{n+1} = \frac{\boldsymbol{s}_{n+1}^{\text{trial}}}{\|\boldsymbol{s}_{n+1}^{\text{trial}}\|} = \frac{\boldsymbol{\varepsilon}_{\mathrm{d}\ n+1}^{e\ \text{trial}}}{\|\boldsymbol{\varepsilon}_{\mathrm{d}\ n+1}^{e\ \text{trial}}\|}$$

…further, we differentiate the return mapping equation

$$\tilde{\Phi}(\Delta\gamma) \equiv q_{n+1}^{\text{trial}} - 3G\,\Delta\gamma - \sigma_y(\bar{\varepsilon}_n^p + \Delta\gamma) = 0$$

and obtain

$$\begin{aligned}\frac{\partial \Delta\gamma}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}} &= \frac{1}{3G+H}\,\frac{\partial q_{n+1}^{\text{trial}}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\ \text{trial}}} \\ &= \frac{2G}{3G+H}\sqrt{\tfrac{3}{2}}\,\bar{\boldsymbol{N}}_{n+1}\end{aligned} \qquad H \equiv \left.\frac{\mathrm{d}\sigma_y}{\mathrm{d}\bar{\varepsilon}^p}\right|_{\bar{\varepsilon}_n^p + \Delta\gamma}$$

Finally, we get the closed form expression for the consistent tangent

$$\boxed{\begin{aligned}\mathsf{D}^{ep} &= \mathsf{D}^{e} - \frac{\Delta\gamma\, 6G^2}{q_{n+1}^{\text{trial}}}\,\mathsf{I}_{\mathrm{d}} + 6G^2\left(\frac{\Delta\gamma}{q_{n+1}^{\text{trial}}} - \frac{1}{3G+H}\right)\bar{\boldsymbol{N}}_{n+1}\otimes\bar{\boldsymbol{N}}_{n+1} \\ &= 2G\left(1 - \frac{\Delta\gamma\, 3G}{q_{n+1}^{\text{trial}}}\right)\mathsf{I}_{\mathrm{d}} \\ &\quad + 6G^2\left(\frac{\Delta\gamma}{q_{n+1}^{\text{trial}}} - \frac{1}{3G+H}\right)\bar{\boldsymbol{N}}_{n+1}\otimes\bar{\boldsymbol{N}}_{n+1} + K\,\boldsymbol{I}\otimes\boldsymbol{I}.\end{aligned}}$$

Solution of the incremental boundary value problem

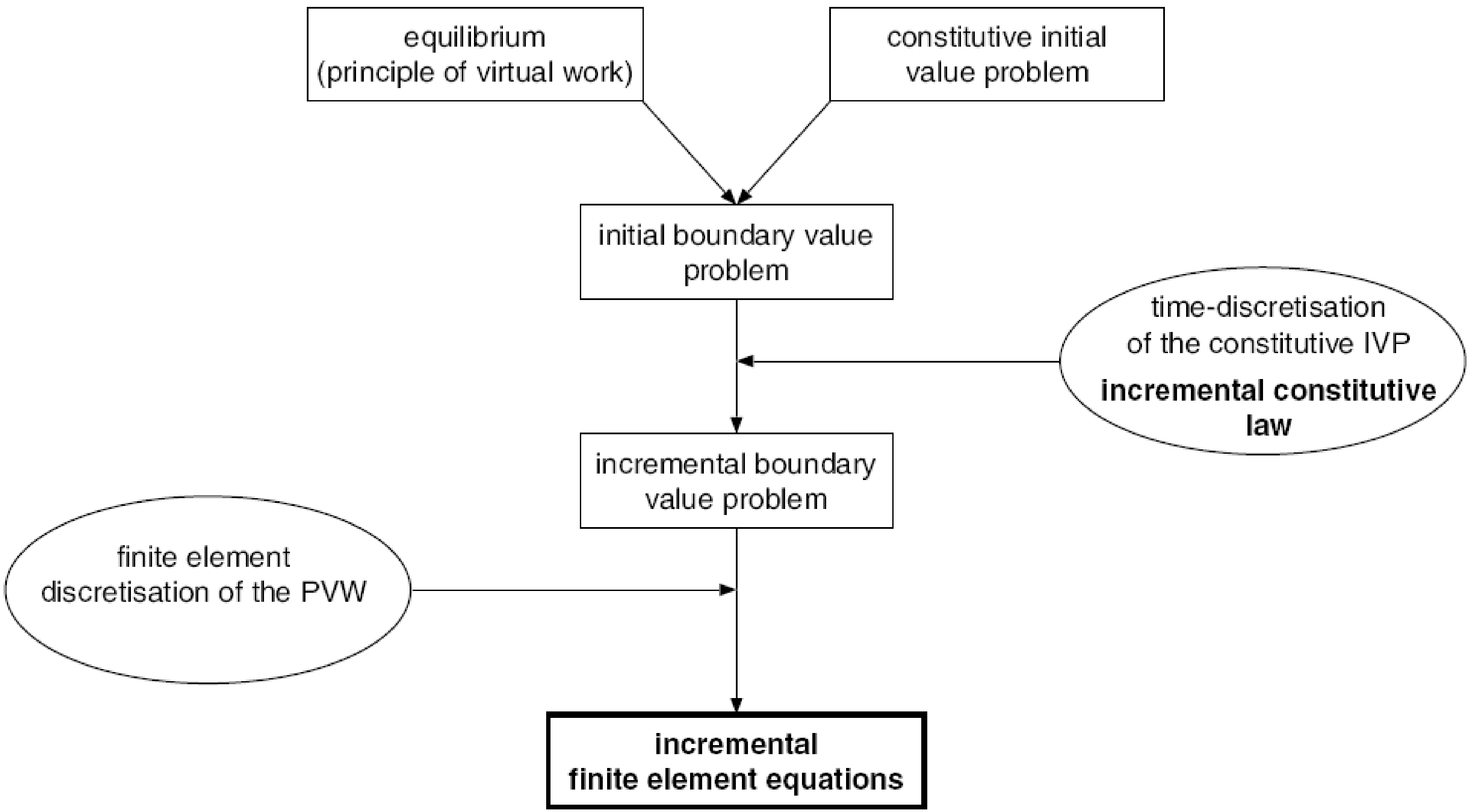

Figure 4.1. Numerical approximations. Reducing the initial boundary value problem to a set of incremental finite element equations.

Incremental boundary value problem. Principle of Virtual Work

$$\int_{\Omega} [\hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \nabla^s \boldsymbol{u}_{n+1}) : \nabla^s \boldsymbol{\eta} - \boldsymbol{b}_{n+1} \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\partial\Omega_t} \boldsymbol{t}_{n+1} \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0$$

Finite element-discretised IBVP

$$\mathbf{r}(\mathbf{u}_{n+1}) \equiv \mathbf{f}^{\,\mathrm{int}}(\mathbf{u}_{n+1}) - \mathbf{f}^{\,\mathrm{ext}}_{\,n+1} = \mathbf{0}$$

$$\mathbf{f}^{\,\mathrm{int}}_{(e)} = \int_{\Omega^{(e)}} \mathbf{B}^T \, \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \boldsymbol{\varepsilon}(\mathbf{u}_{n+1})) \, \mathrm{d}v$$

$$\mathbf{f}^{\,\mathrm{ext}}_{(e)} = \int_{\Omega^{(e)}} \mathbf{N}^T \, \boldsymbol{b}_{n+1} \, \mathrm{d}v + \int_{\partial\Omega_t^{(e)}} \mathbf{N}^T \, \boldsymbol{t}_{n+1} \, \mathrm{d}a$$

Newton-Raphson iterative solution

$$\mathbf{K}_T\, \delta\mathbf{u}^{(k)} = -\mathbf{r}^{(k-1)}$$

$$\mathbf{r}^{(k-1)} \equiv \mathbf{f}^{\,\mathrm{int}}(\mathbf{u}_{n+1}^{(k-1)}) - \mathbf{f}_{\,n+1}^{\,\mathrm{ext}}$$

$$\boxed{\mathbf{K}_T \equiv \int_{{}^h\Omega} (\mathbf{B}^g)^T \mathrm{D}\; \mathbf{B}^g \, dv = \left.\frac{\partial \mathbf{r}}{\partial \mathbf{u}_{n+1}}\right|_{\mathbf{u}_{n+1}^{(k-1)}}}$$

$$\boxed{\mathsf{D} \equiv \left.\frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right|_{\boldsymbol{\varepsilon}_{n+1}^{(k-1)}}}$$

$$\mathbf{u}_{n+1}^{(k)} = \mathbf{u}_{n+1}^{(k-1)} + \delta\mathbf{u}^{(k)}$$

Convergence criterion

$$\frac{\left|\mathbf{r}^{(m)}\right|}{\left|\mathbf{f}_{\,n+1}^{\,\mathrm{ext}}\right|} \leq \epsilon_{\mathrm{tol}}$$

Newton-Raphson iterative solution

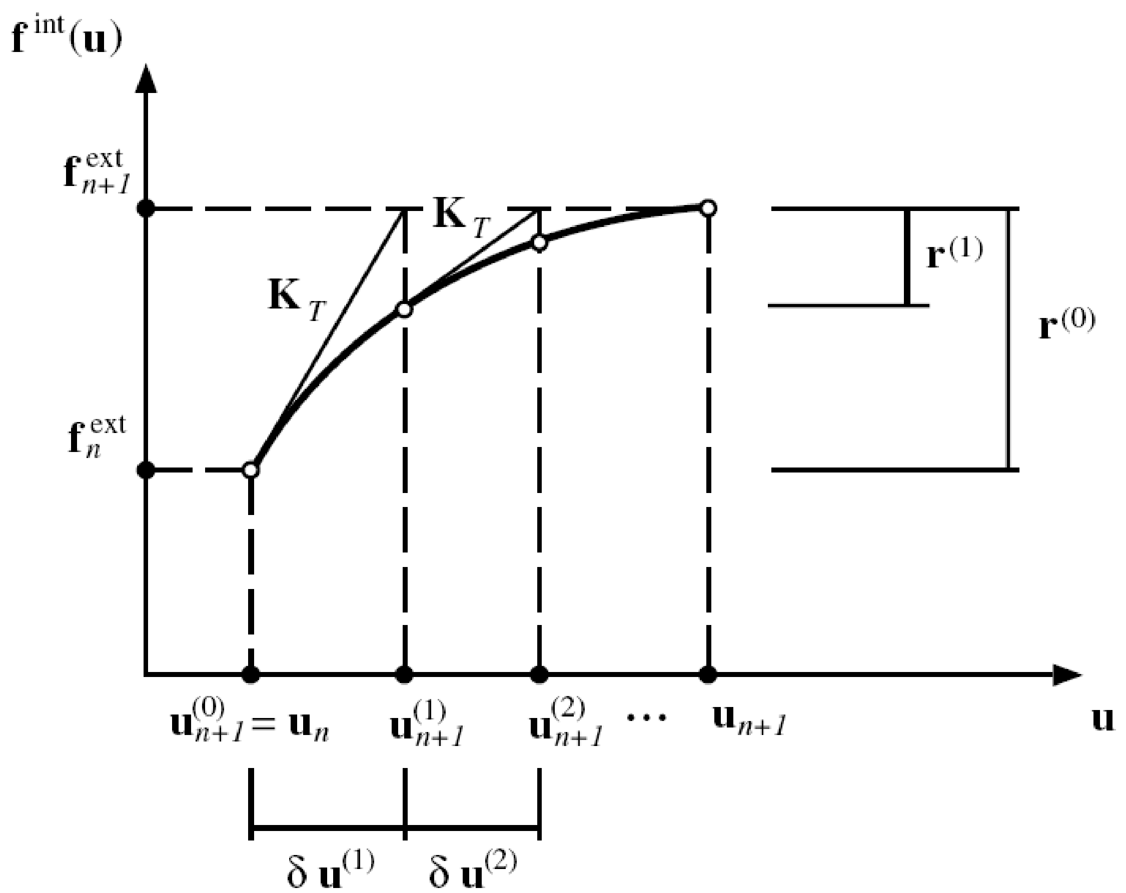

Figure 4.6. The Newton–Raphson algorithm for the incremental finite element equilibrium equation.

Box 4.2. The Newton–Raphson scheme for solution of the incremental nonlinear finite element equation (infinitesimal strains).

(i) $k := 0$. Set initial guess and residual

$$\mathbf{u}_{n+1}^{(0)} := \mathbf{u}_n; \qquad \mathbf{r} := \mathbf{f}^{\,\text{int}}(\mathbf{u}_n) - \lambda_{n+1}\bar{\mathbf{f}}^{\,\text{ext}}$$

(ii) Compute consistent tangent matrices [MATICT]

$$\mathbf{D} := \partial\hat{\boldsymbol{\sigma}}/\partial\boldsymbol{\varepsilon}_{n+1}$$

(iii) Assemble element tangent stiffness matrices [ELEIST, STSTD2]

$$\mathbf{K}_T^{(e)} := \sum_{i=1}^{n_{\text{gausp}}} w_i\; j_i\; \mathbf{B}_i^T\; \mathbf{D}_i\; \mathbf{B}_i$$

(iv) $k := k + 1$. Assemble global stiffness and solve for $\delta\mathbf{u}^{(k)}$ [FRONT]

$$\mathbf{K}_T\;\delta\mathbf{u}^{(k)} = -\mathbf{r}^{(k-1)}$$

(v) Apply Newton correction to displacements [UPCONF]

$$\mathbf{u}_{n+1}^{(k)} := \mathbf{u}_{n+1}^{(k-1)} + \delta\mathbf{u}^{(k)}$$

(vi) Update strains [IFSTD2]

$$\boldsymbol{\varepsilon}_{n+1}^{(k)} := \mathbf{B}\;\mathbf{u}_{n+1}^{(k)}$$

(vii) Use constitutive integration algorithm to update stresses and other state variables [MATISU]

$$\boldsymbol{\sigma}_{n+1}^{(k)} := \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \boldsymbol{\varepsilon}_{n+1}^{(k)}); \qquad \boldsymbol{\alpha}_{n+1}^{(k)} := \hat{\boldsymbol{\alpha}}(\boldsymbol{\alpha}_n, \boldsymbol{\varepsilon}_{n+1}^{(k)})$$

(viii) Compute element internal force vectors [INTFOR, IFSTD2]

$$\mathbf{f}_{(e)}^{\,\text{int}} := \sum_{i=1}^{n_{\text{gausp}}} w_i\; j_i\; \mathbf{B}_i^T\;\; \boldsymbol{\sigma}_{n+1}^{(k)}\Big|_i$$

(ix) Assemble global internal force vector and update residual [CONVER]

$$\mathbf{r} := \mathbf{f}^{\,\text{int}} - \lambda_{n+1}\bar{\mathbf{f}}^{\,\text{ext}}$$

(x) Check for convergence [CONVER]

IF $\|\mathbf{r}\|/\|\mathbf{f}^{\,\text{ext}}\| \leq \epsilon_{\text{tol}}$ THEN set $(\cdot)_{n+1} := (\cdot)_{n+1}^{(k)}$ and EXIT

ELSE GOTO (ii)

Example. Pressurised cylinder

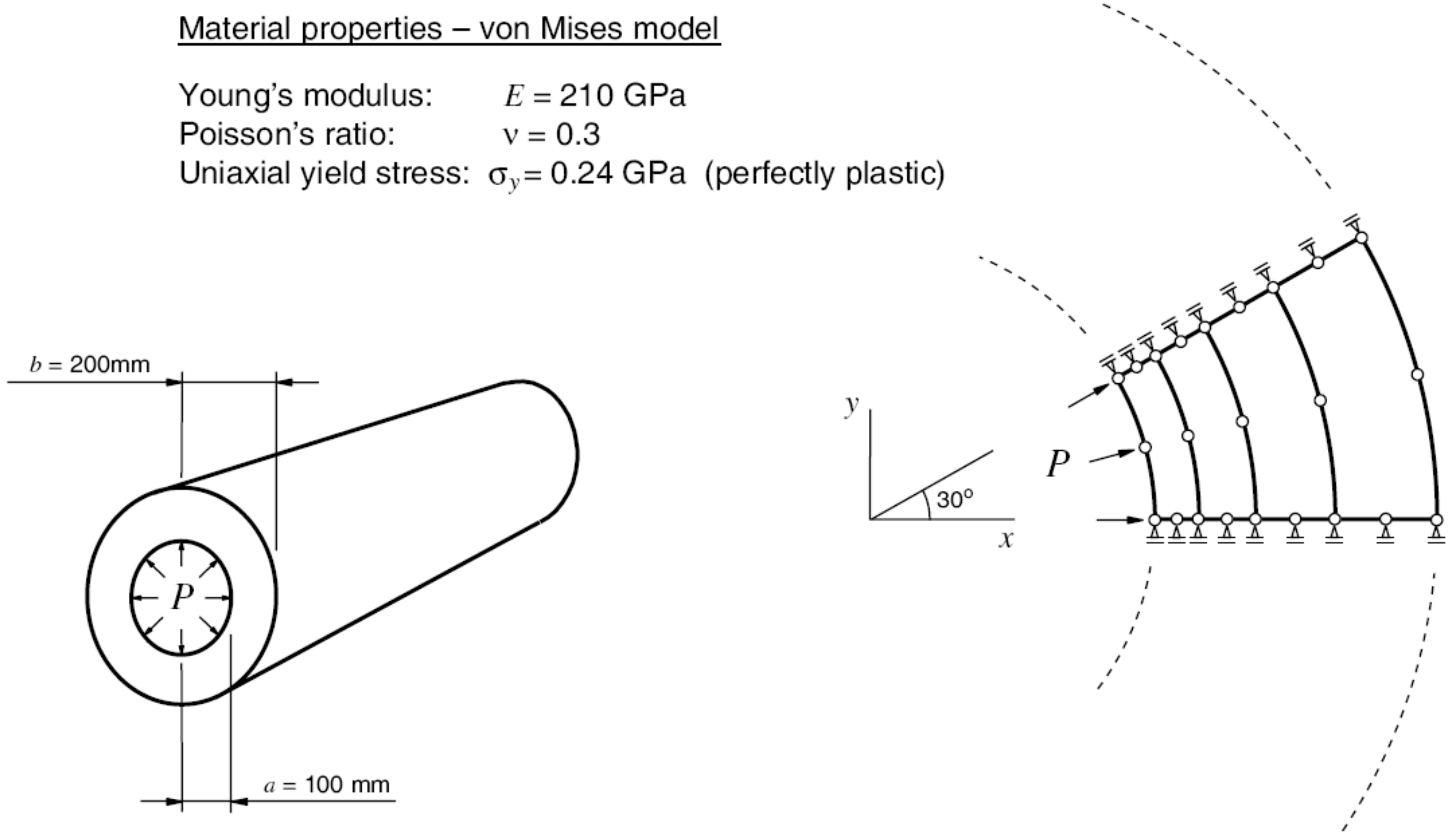

Figure 7.14. Internally pressurised cylinder. Geometry, material properties and finite element mesh.

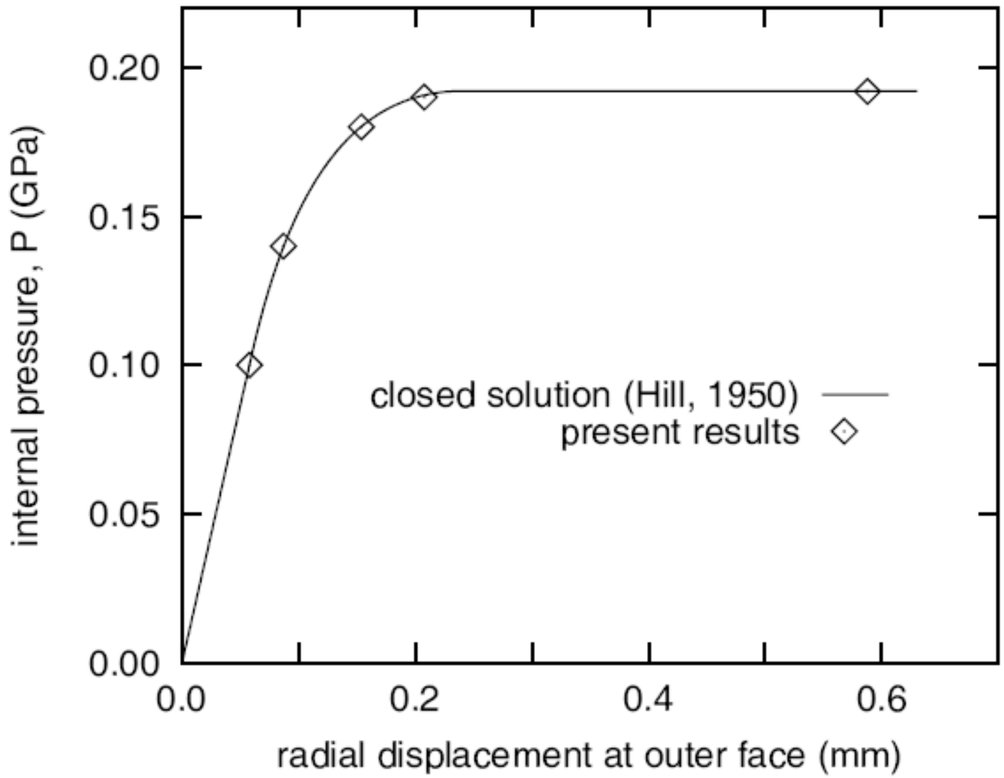

Figure 7.16. Internally pressurised cylinder. Pressure versus displacement diagram.

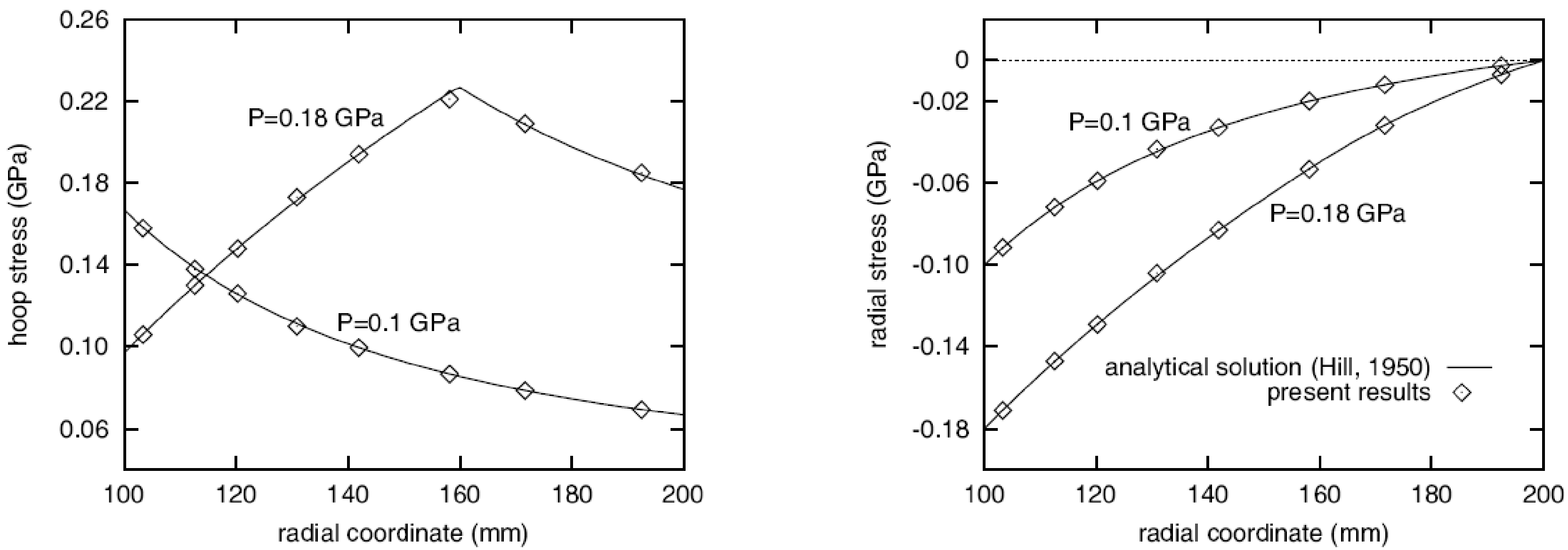

Figure 7.17. Internally pressurised cylinder. Hoop and radial stress distributions at different levels of applied internal pressure. Finite element results are computed at Gauss integration points.

…quadratic convergence in equilibrium problem solution