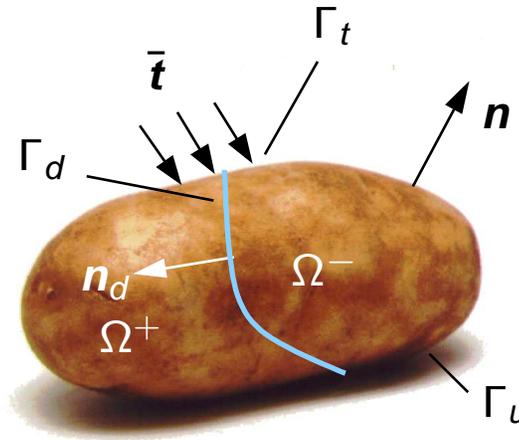# GFEM for Three-Dimensional Fractures

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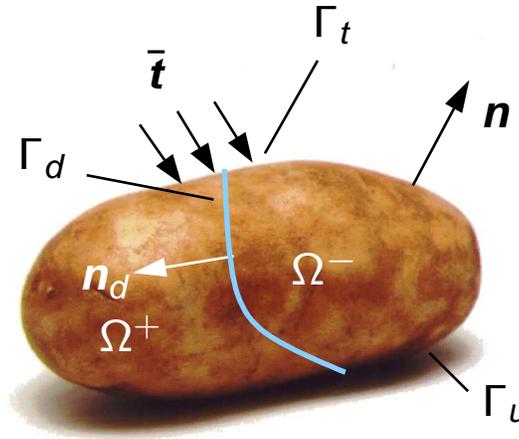
caduarte@illinois.edu

Governing Equations: 3-D Elasticity

- ▶ equilibrium $\qquad \nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad$ in Ω

- ▶ traction bc $\qquad \boldsymbol{\sigma} \boldsymbol{n} = \bar{\boldsymbol{t}} \quad$ on Γ_t

- ▶ traction bc on fracture faces $\quad \boldsymbol{\sigma} \boldsymbol{n}_d = \bar{\boldsymbol{t}}_d \quad$ on Γ_d

- ▶ displacement bc $\qquad \boldsymbol{u} = \bar{\boldsymbol{u}} \quad$ on Γ_u

- ▶ kinematics $\qquad \boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u}$ in Ω

- ▶ constitutive relation in the bulk $\qquad \boldsymbol{\sigma} = \boldsymbol{E} : \boldsymbol{\varepsilon}$ in Ω

Governing Equations: 3-D Elasticity

- ▶ Weak Formulation

- ▶ Find $\boldsymbol{u}^{hp} \in \boldsymbol{X}(\Omega) \subset (H^1(\Omega))^3$, such that $\forall \, \boldsymbol{v}^{hp} \in \boldsymbol{X}(\Omega)$

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}^{hp}) : \boldsymbol{\varepsilon}(\boldsymbol{v}^{hp}) d\boldsymbol{x} + \eta \int_{\Gamma_u} \boldsymbol{u}^{hp} \cdot \boldsymbol{v}^{hp} d\boldsymbol{s}$$

$$= \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}^{hp} d\boldsymbol{s} + \int_{\Gamma_d^+} \bar{\boldsymbol{t}}_d^+ \cdot [\![\boldsymbol{v}^{hp}]\!] d\boldsymbol{s} + \eta \int_{\Gamma_u} \bar{\boldsymbol{u}} \cdot \boldsymbol{v}^{hp} d\boldsymbol{s}$$

where $\boldsymbol{X}(\Omega)$ is a GFEM discretization of $(H^1(\Omega))^3$ and η is a penalty parameter

GFEM Approximation $u^{hp} \in X$

$$u^{hp}(x) = \sum_{\alpha=1}^{N} \sum_{j=1}^{D_L} \underline{u}_{\alpha i} \phi_{\alpha i}(x)$$

$$\begin{bmatrix} u^{hp} \\ v^{hp} \\ w^{hp} \end{bmatrix} = \sum_{\alpha=1}^{N} \sum_{j=1}^{D_L} \begin{bmatrix} \phi_{\alpha j}(x) & 0 & 0 \\ 0 & \phi_{\alpha j}(x) & 0 \\ 0 & 0 & \phi_{\alpha j}(x) \end{bmatrix} \begin{bmatrix} u_{\alpha j} \\ v_{\alpha j} \\ w_{\alpha j} \end{bmatrix} = N\underline{u}$$

GFEM Approximation $u^{hp} \in X$

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▶ using Voigt notation to write ε

$$\underline{\varepsilon}(u^{hp}) = DN\underline{u} = B\underline{u}$$

where D is the differential operator ∇^s in matrix format.

▶ similarly for virtual displacements

$$v^{hp}(x) = N\underline{v}$$

$$\underline{\varepsilon}(v^{hp}) = DN\underline{v} = B\underline{v}$$

Discrete Equations

- Plugging the GFEM approximation in the weak form leads to

$$B(\boldsymbol{u}^{hp}, \boldsymbol{v}^{hp}) \;=\; \underline{\boldsymbol{v}}^T \int_\Omega \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} dx \, \underline{\boldsymbol{u}} + \eta \underline{\boldsymbol{v}}^T \int_{\Gamma_u} \boldsymbol{N}^T \boldsymbol{N} ds \, \underline{\boldsymbol{u}}$$

$$=\; \underline{\boldsymbol{v}}^T \boldsymbol{K} \underline{\boldsymbol{u}} + \eta \underline{\boldsymbol{v}}^T \boldsymbol{M}_s \, \underline{\boldsymbol{u}}$$

$$L(\boldsymbol{v}^{hp}) \;=\; \underline{\boldsymbol{v}}^T \int_{\Gamma_t} \boldsymbol{N}^T \bar{\boldsymbol{t}} ds + \underline{\boldsymbol{v}}^T \int_{\Gamma_d^+} \bar{\boldsymbol{t}}_d^+ [\![\boldsymbol{N}]\!]^T ds + \eta \underline{\boldsymbol{v}}^T \int_{\Gamma_u} \boldsymbol{N}^T \bar{\boldsymbol{u}} ds$$

$$=\; \underline{\boldsymbol{v}}^T \boldsymbol{F}_t + \underline{\boldsymbol{v}}^T \boldsymbol{F}_d + \eta \underline{\boldsymbol{v}}^T \boldsymbol{F}_s$$

- This leads to

$$(\boldsymbol{K} + \eta \boldsymbol{M}_s) \, \underline{\boldsymbol{u}} = \boldsymbol{F}_t + \boldsymbol{F}_d + \eta \boldsymbol{F}_s$$

GFEM Approximation for 3-D Fractures

- A GFEM approximation $u^{hp}(x) \in X(\Omega)$ of a vector value function u is given by

$$u^{hp}(x) = \sum_{\alpha=1}^{N} \varphi_\alpha(x) u_\alpha^{hp}(x)$$

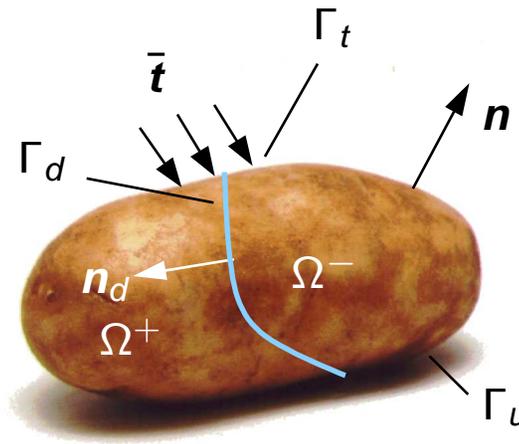where $u_\alpha^{hp}(x) \in \chi_\alpha$ the space defined on cloud or patch ω_α

$$\chi_\alpha = \text{span}\{L_{\alpha j}, \ 1 \leq j \leq D_L(\alpha), \ L_{\alpha j} \in H^1(\omega_\alpha)\}.$$

- Goal: Select basis functions for χ_α such that

$$\inf_{w_\alpha \in \chi_{\alpha(\omega_\alpha)}} \|u - w_\alpha\|_{\varepsilon(\omega_\alpha)} < \epsilon$$

- The optimal selection depends on the behavior of the exact solution u on ω_α: Use a-priori knowledge about u.

GFEM Approximation for 3-D Fractures

▶ In the case of fracture mechanics problems, the elasticity solution $\boldsymbol{u}$ may be decomposed into
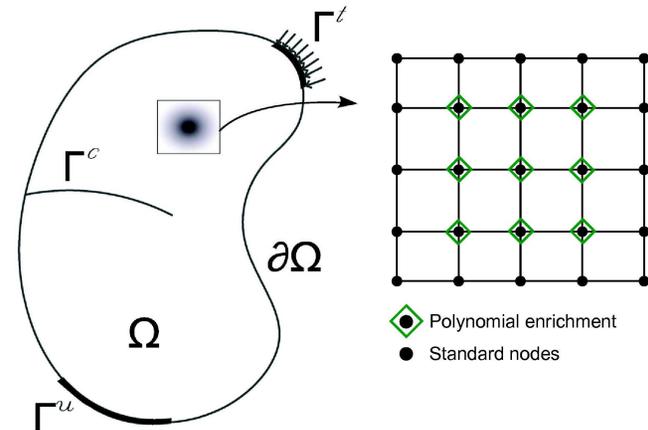
$$\boldsymbol{u} = \hat{\boldsymbol{u}} + \tilde{\tilde{\boldsymbol{u}}} + \breve{\boldsymbol{u}}$$

where $\hat{\boldsymbol{u}}$ is a continuous function, $\tilde{\tilde{\boldsymbol{u}}}$ is a discontinuous function but non-singular and $\breve{\boldsymbol{u}}$ is a discontinuous and singular function.

Enrichments for clouds away from fracture

- Cloud approximation $\hat{\boldsymbol{u}}_\alpha^{hp}(\boldsymbol{x})$ of $\boldsymbol{u}$ over ω_α

$$\boldsymbol{u}_\alpha^{hp}(\boldsymbol{x}) = \hat{\boldsymbol{u}}_\alpha^{hp}(\boldsymbol{x}) = \sum_{j=1}^{\hat{D}_L} \underline{\hat{\boldsymbol{u}}}_{\alpha j} \hat{L}_{\alpha j}(\boldsymbol{x})$$

where $\hat{D}_L$ is the dimension of a set of polynomial enrichment functions of degree less than or equal to $p-1$

$$\left\{ \hat{L}_{\alpha j} \right\}_{j=1}^{\hat{D}_L} = \left\{ 1, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \frac{(z - z_\alpha)}{h_\alpha}, \frac{(x - x_\alpha)^2}{h_\alpha^2}, \frac{(y - y_\alpha)^2}{h_\alpha^2}, \cdots \right\}$$
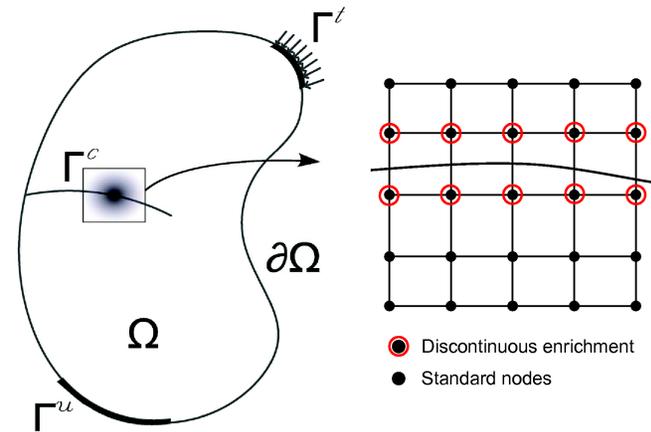
with h_α being a scaling factor.

[Duarte, Babuška & Oden, CS 2000;
Oden, Duarte & Zienkiewicz, CMAME 1998]

▶ elasticity solution over cloud ω_α cut by the fracture but not the front

$$\boldsymbol{u} = \hat{\boldsymbol{u}} + \tilde{\hat{\boldsymbol{u}}} = \hat{\boldsymbol{u}} + \mathcal{H}\tilde{\boldsymbol{u}}$$

$$\mathcal{H}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \Omega^+ \\ 0 & \text{otherwise} \end{cases}$$

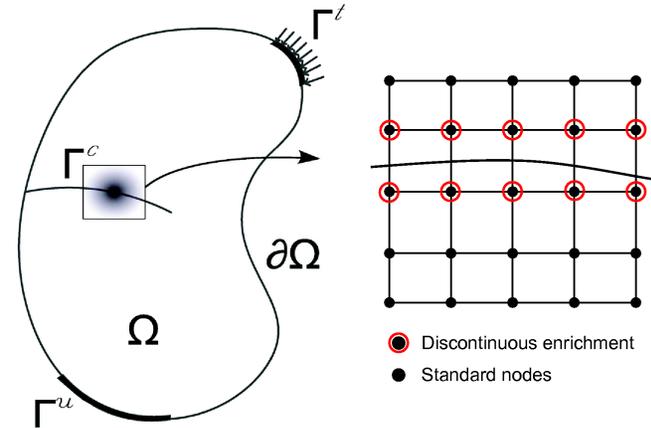○ Discontinuous enrichment
● Standard nodes

Enrichments for Clouds Cut by Fracture

- elasticity solution over cloud ω_α cut by the fracture but not the front

$$\boldsymbol{u} = \hat{\boldsymbol{u}} + \tilde{\tilde{\boldsymbol{u}}} = \hat{\boldsymbol{u}} + \mathcal{H}\tilde{\boldsymbol{u}}$$

$$\mathcal{H}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \Omega^+ \\ 0 & \text{otherwise} \end{cases}$$

○ Discontinuous enrichment
● Standard nodes

- cloud approximation over ω_α

$$\boldsymbol{u}_\alpha^{hp}(\boldsymbol{x}) = \hat{\boldsymbol{u}}_\alpha^{hp}(\boldsymbol{x}) + \mathcal{H}\tilde{\boldsymbol{u}}_\alpha^{hp}(\boldsymbol{x}) = \sum_{j=1}^{\hat{D}_L} \hat{\underline{\boldsymbol{u}}}_{\alpha j}\hat{L}_{\alpha j}(\boldsymbol{x}) + \sum_{j=1}^{\tilde{D}_L} \tilde{\underline{\boldsymbol{u}}}_{\alpha j}\mathcal{H}\hat{L}_{\alpha j}(\boldsymbol{x})$$

- GFEM shape functions at these nodes (we take $\tilde{D}_L = \hat{D}_L$)

$$\varphi \times \left\{ 1, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \ldots, \mathcal{H}, \mathcal{H}\frac{(x - x_\alpha)}{h_\alpha}, \mathcal{H}\frac{(y - y_\alpha)}{h_\alpha}, \ldots \right\}$$
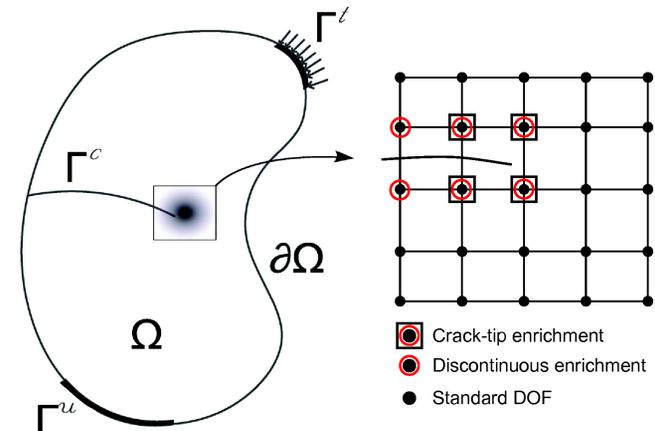
Enrichments for Clouds Cut by Fracture
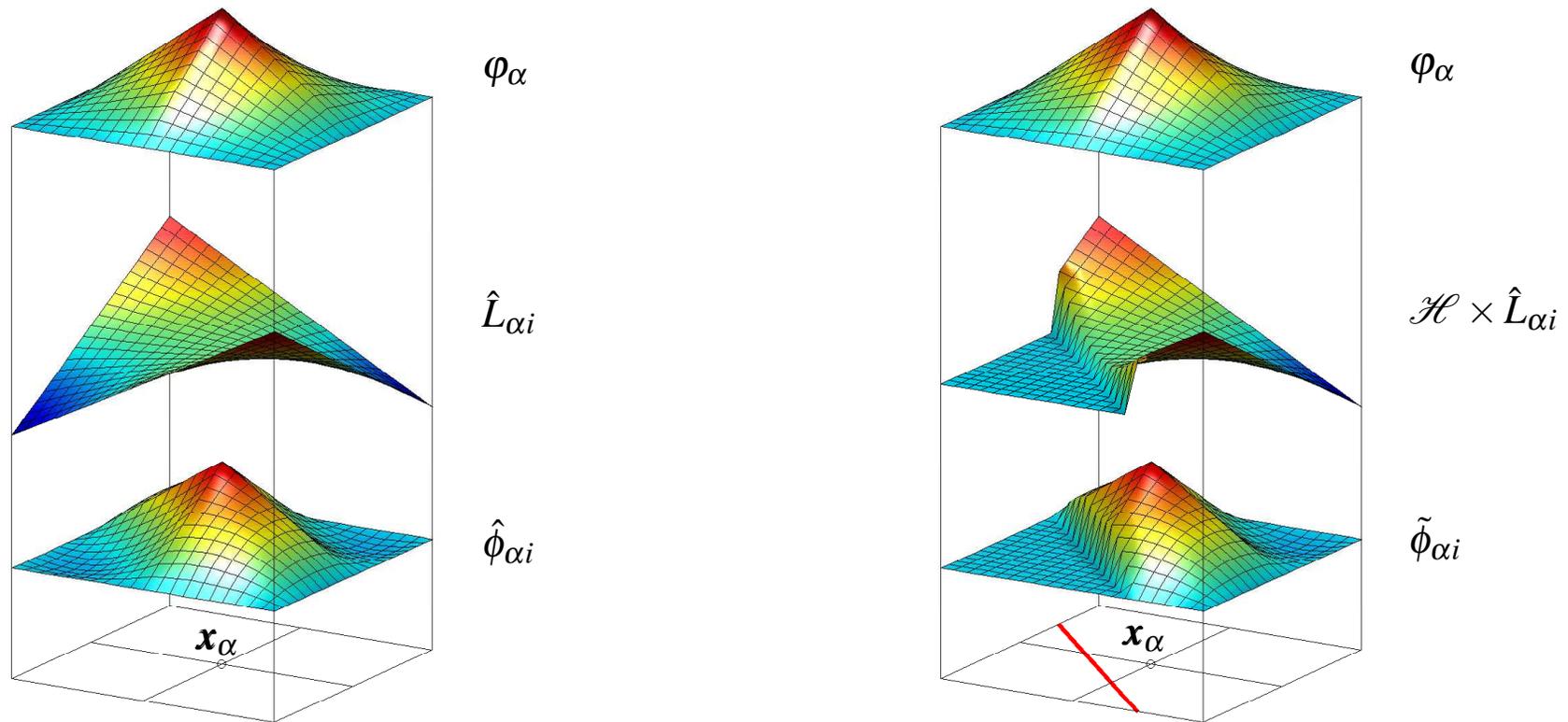
(a) Continuous high-order enrichment

(b) Discontinuous high-order enrichment

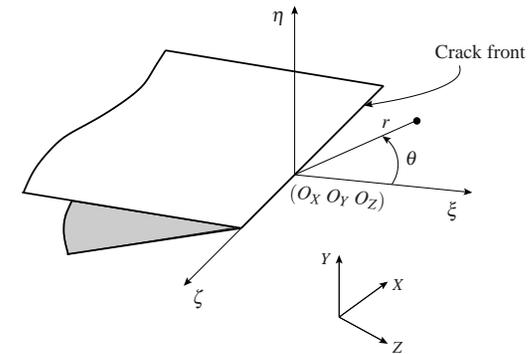Figure: Construction of a GFEM shape function using continuous and discontinuous polynomial enrichments.
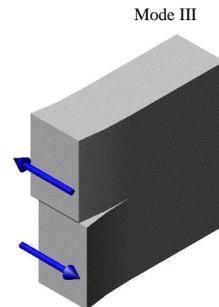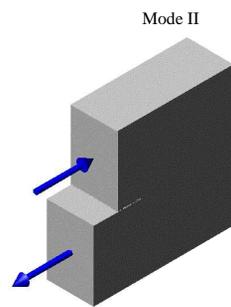
Enrichments for Clouds Cut by Fracture Front

▶ Cloud approximation $\breve{u}^{hp}_\alpha(\boldsymbol{x})$ of $\boldsymbol{u}$ over ω_α is based on 2-D expansions of the elasticity solution near a fracture front

▶ fracture front can be anywhere

▶ more accurate than Heaviside function

Γ^t

Γ^c

$\partial\Omega$

Ω

Γ^u

◉ Crack-tip enrichment
◉ Discontinuous enrichment
● Standard DOF

Enrichments for Clouds Cut by Fracture Front

▶ expansions of elasticity solution near a fracture front
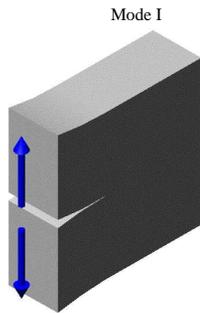
$$u_\xi^{(I)}(r,\theta) = \frac{\sqrt{r}}{2G}\left\{\left[\kappa - \frac{1}{2}\right]\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3}{2}\theta\right\}$$

$$u_\eta^{(I)}(r,\theta) = \frac{\sqrt{r}}{2G}\left\{\left[\kappa + \frac{1}{2}\right]\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3}{2}\theta\right\} \qquad u_\zeta^{(I)}(r,\theta) = 0$$

Enrichments for Clouds Cut by Fracture Front

- ▶ select basis for cloud space from this expansion

- ▶ this leads to so-called Oden-Duarte (OD) enrichments:

$$
\check{\boldsymbol{u}}_\alpha^{hp} = \sum_{j=1}^{2}
\begin{bmatrix}
\check{L}^\xi_{\alpha j}(r,\theta) & 0 & 0 \\
0 & \check{L}^\eta_{\alpha j}(r,\theta) & 0 \\
0 & 0 & \check{L}^\zeta_{\alpha j}(r,\theta)
\end{bmatrix}
\begin{bmatrix}
\underline{\check{u}}^\xi_{\alpha j} \\
\underline{\check{u}}^\eta_{\alpha j} \\
\underline{\check{u}}^\zeta_{\alpha j}
\end{bmatrix}
$$

$$
\check{L}^\xi_{\alpha 1}(r,\theta) = u^{(I)}_\xi(r,\theta), \quad
\check{L}^\eta_{\alpha 1}(r,\theta) = u^{(I)}_\eta(r,\theta), \quad
\check{L}^\zeta_{\alpha 1}(r,\theta) = u^{(III)}_\zeta(r,\theta)
$$

- ▶ these are vector-valued enrichments: Different enrichments for each direction

- ▶ they must be transformed to global coordinates (simple rotation)

[Oden & Duarte, MAFELAP 1996; Duarte et al., CS 2000]

Enrichments for Clouds Cut by Fracture Front

- ▶ OD enrichments are equivalent to a more recent enrichment, the so-called Belytschko-Black (BB) enrichments

[Fleming, Chu, Moran & Belytschko, IJNME 1997
Belytschko & Black, IJNME 1999]:

$$\breve{\boldsymbol{u}}_{\alpha}^{hp} = \underline{\breve{\boldsymbol{u}}}_{\alpha 1} \sqrt{r} \sin \frac{\theta}{2} + \underline{\breve{\boldsymbol{u}}}_{\alpha 2} \sqrt{r} \cos \frac{\theta}{2}$$
$$+ \underline{\breve{\boldsymbol{u}}}_{\alpha 3} \sqrt{r} \sin \frac{\theta}{2} \sin \theta + \underline{\breve{\boldsymbol{u}}}_{\alpha 4} \sqrt{r} \cos \frac{\theta}{2} \sin \theta$$

BB and OD are equivalent: different basis of the same space

Enrichments for Clouds Cut by Fracture Front

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$$\breve{\boldsymbol{u}}_{\alpha}^{hp} = \underline{\breve{\boldsymbol{u}}}_{\alpha 1}\sqrt{r}\sin\frac{\theta}{2} + \underline{\breve{\boldsymbol{u}}}_{\alpha 2}\sqrt{r}\cos\frac{\theta}{2}$$

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BB and OD are equivalent: different basis of the same space

BB or OD?

Enrichments for Clouds Cut by Fracture Front

- ► OD enrichments are equivalent to a more recent enrichment, the so-called Belytschko-Black (BB) enrichments

[Fleming, Chu, Moran & Belytschko, IJNME 1997
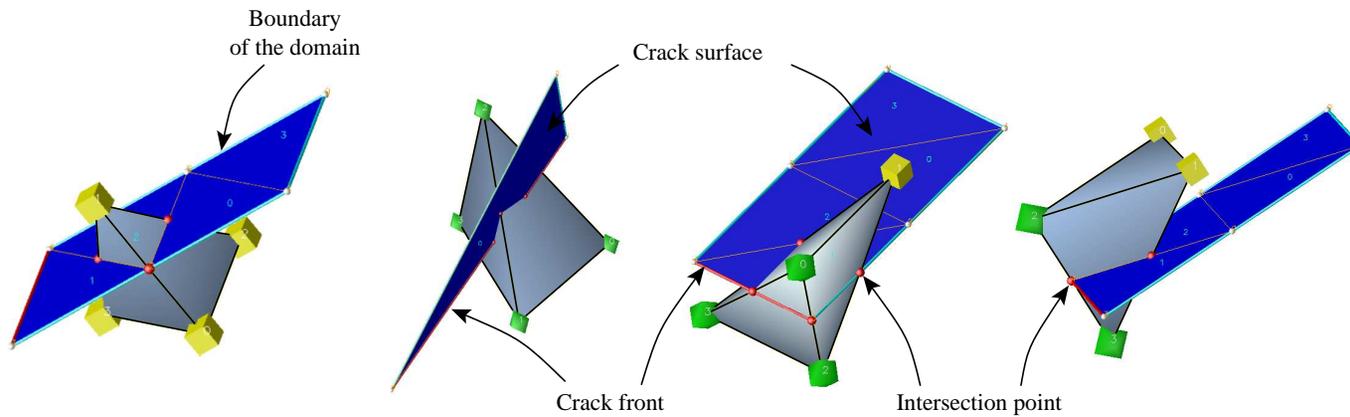Belytschko & Black, IJNME 1999]:

$$\breve{\boldsymbol{u}}_\alpha^{hp} = \underline{\breve{\boldsymbol{u}}}_{\alpha 1} \sqrt{r} \sin \frac{\theta}{2} + \underline{\breve{\boldsymbol{u}}}_{\alpha 2} \sqrt{r} \cos \frac{\theta}{2}$$
$$+ \underline{\breve{\boldsymbol{u}}}_{\alpha 3} \sqrt{r} \sin \frac{\theta}{2} \sin \theta + \underline{\breve{\boldsymbol{u}}}_{\alpha 4} \sqrt{r} \cos \frac{\theta}{2} \sin \theta$$

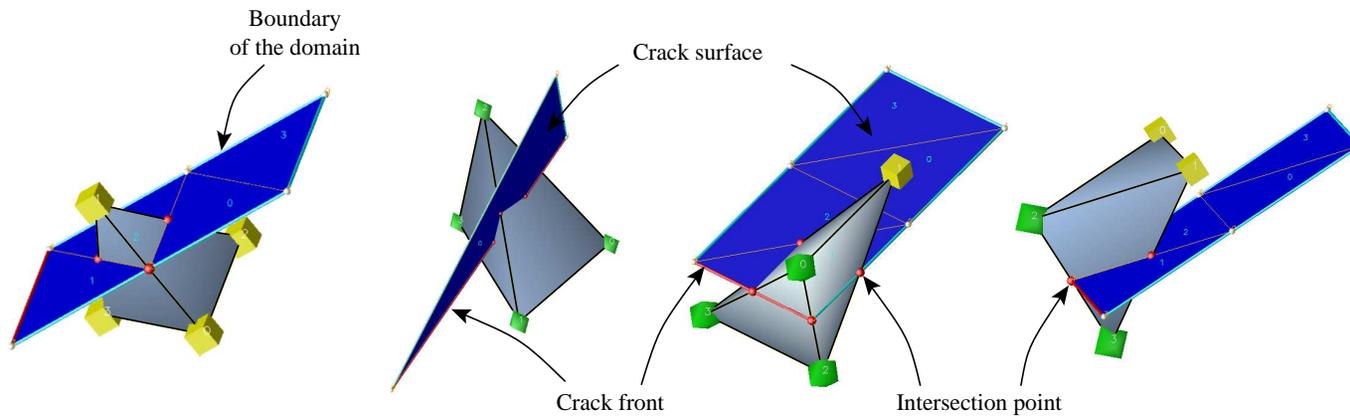BB and OD are equivalent: different basis of the same space

BB or OD?

- ► OD can be implemented using fewer dofs: 6 in 3-D and 4 in 2-D enrichment dofs per node

- ► BB leads to 12 enrichment dofs per node in 3-D and 8 enrichment dofs per node in 2-D

Examples

Examples of elements cut by a crack surface

Examples

Examples of elements cut by a crack surface
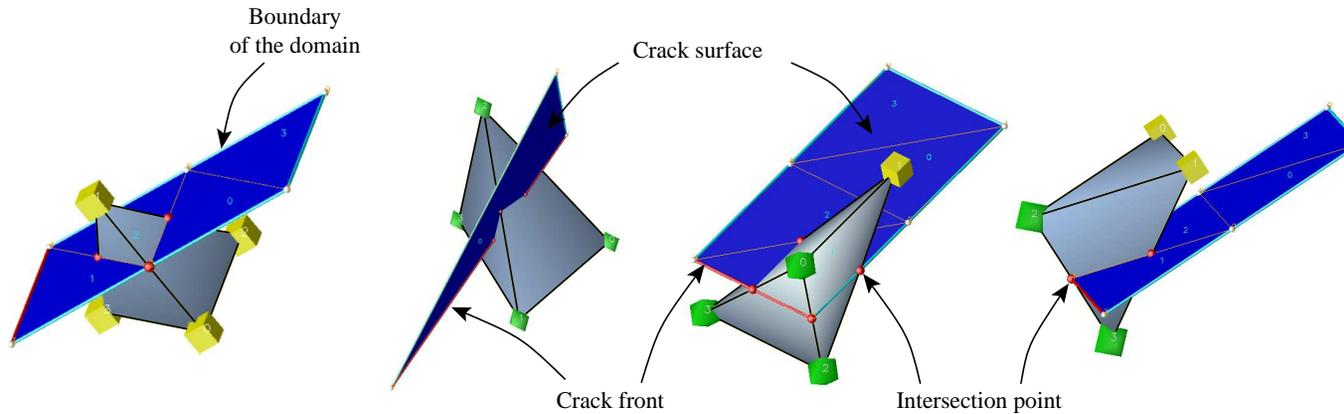
- *Linear Approximation:*

The shape functions at a yellow node $\boldsymbol{x}_\alpha$ are given by

$$\varphi_\alpha \times \{1, \mathcal{H}\}$$

At a green node $\boldsymbol{x}_\alpha$, the shape functions for the $x-$, $y-$ and $z-$ components of the displacement vector are given by

$$\varphi_\alpha \times \left\{1, \check{L}^x_{\alpha 1}, \check{L}^x_{\alpha 2}\right\}, \quad \varphi_\alpha \times \left\{1, \check{L}^y_{\alpha 1}, \check{L}^y_{\alpha 2}\right\}, \quad \varphi_\alpha \times \left\{1, \check{L}^z_{\alpha 1}, \check{L}^z_{\alpha 2}\right\}$$

Examples

Boundary of the domain
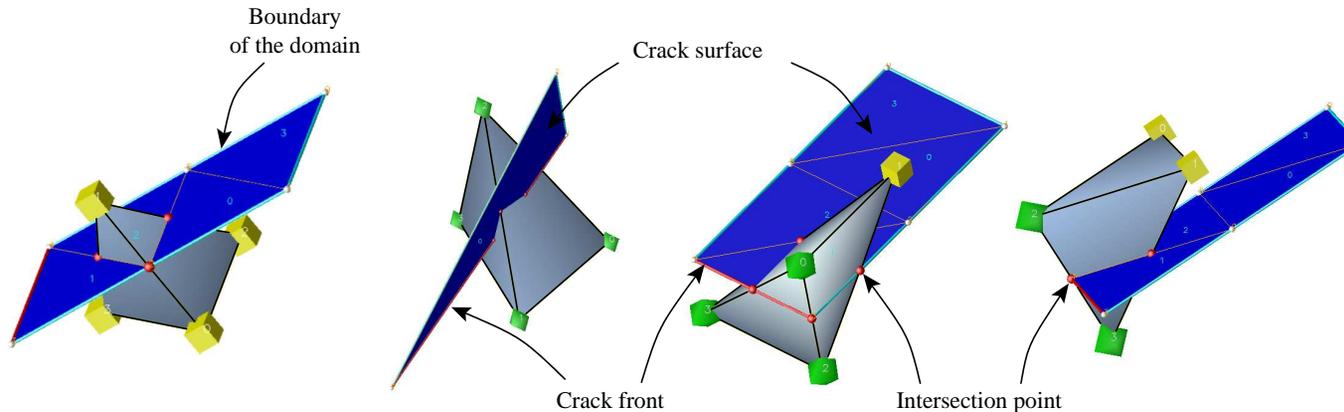
Crack surface

Crack front

Intersection point

Examples of elements cut by a crack surface

▶ *Quadratic Approximation:*

The shape functions at a yellow node $\boldsymbol{x}_\alpha$ are given by

$$
\varphi_\alpha \times \left\{ 1, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \frac{(z - z_\alpha)}{h_\alpha}, \right.
$$
$$
\left. \mathcal{H}, \mathcal{H}\frac{(x - x_\alpha)}{h_\alpha}, \mathcal{H}\frac{(y - y_\alpha)}{h_\alpha}, \mathcal{H}\frac{(z - z_\alpha)}{h_\alpha} \right\}
$$

Examples

Examples of elements cut by a crack surface

At a green node $\boldsymbol{x}_\alpha$, the shape functions for the $x-$, $y-$ and $z-$ components of the displacement vector are given by

$$\varphi_\alpha \times \left\{ 1, \breve{L}^x_{\alpha 1}, \breve{L}^x_{\alpha 2}, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \frac{(z - z_\alpha)}{h_\alpha} \right\},$$

$$\varphi_\alpha \times \left\{ 1, \breve{L}^y_{\alpha 1}, \breve{L}^y_{\alpha 2}, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \frac{(z - z_\alpha)}{h_\alpha} \right\},$$

$$\varphi_\alpha \times \left\{ 1, \breve{L}^z_{\alpha 1}, \breve{L}^z_{\alpha 2}, \frac{(x - x_\alpha)}{h_\alpha}, \frac{(y - y_\alpha)}{h_\alpha}, \frac{(z - z_\alpha)}{h_\alpha} \right\}$$
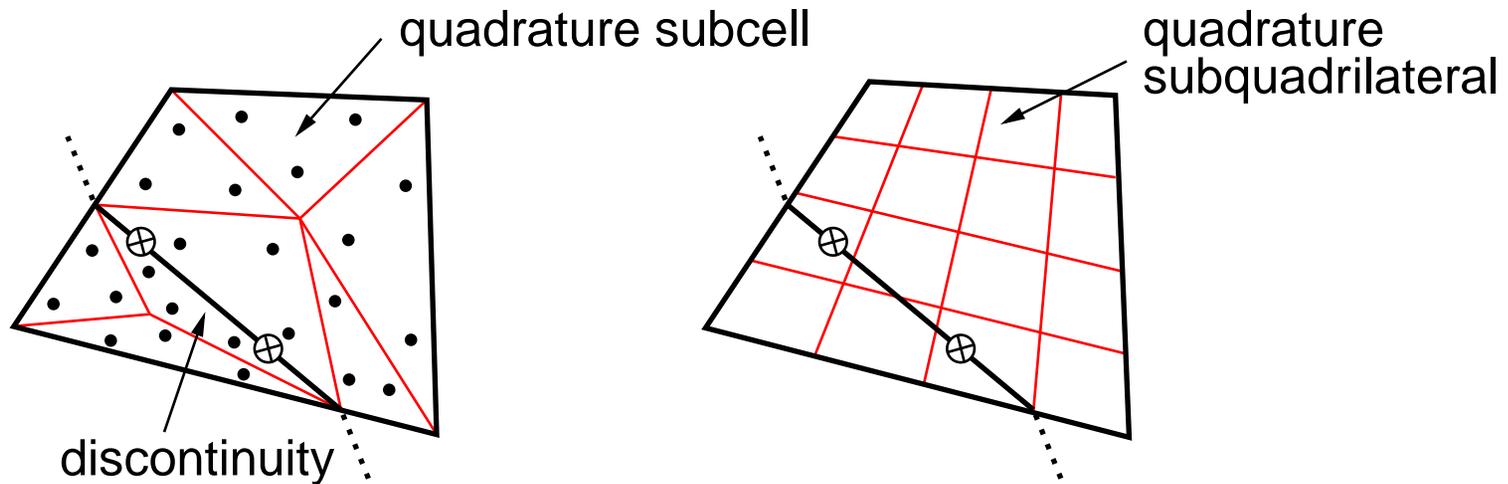
Computational Aspects: Integration of Discrete Equations

$$\int_{\Omega} \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} d\boldsymbol{x} \, \underline{\boldsymbol{u}} + \eta \int_{\Gamma_u} \boldsymbol{N}^T \boldsymbol{N} d\boldsymbol{s} \, \underline{\boldsymbol{u}} =$$

$$\int_{\Gamma_t} \boldsymbol{N}^T \bar{\boldsymbol{t}} d\boldsymbol{s} + \int_{\Gamma_d^+} \bar{\boldsymbol{t}}_d^+ [\![\boldsymbol{N}]\!]^T d\boldsymbol{s} + \eta \int_{\Gamma_u} \boldsymbol{N}^T \bar{\boldsymbol{u}} d\boldsymbol{s}$$

▶ integrand is discontinuous and singular:
  How to integrate it efficiently in 3-D?

Integration of Stiffness Matrix

- ▶ quadrature subcells and subelements

quadrature subcell

quadrature
subquadrilateral

discontinuity

  - ▶ quadrature subcells are tedious to code... but it is the most efficient approach

  - ▶ subcells are also useful for post-processing, computations of enrichment functions, etc.

- ▶ many other approaches proposed: Adaptive integration, equivalent polynomials, special quadrature rules built on the fly, etc....

Finite Element Implementation

what's different compared to standard FEM?

- **preprocessor**

  geometrical engine
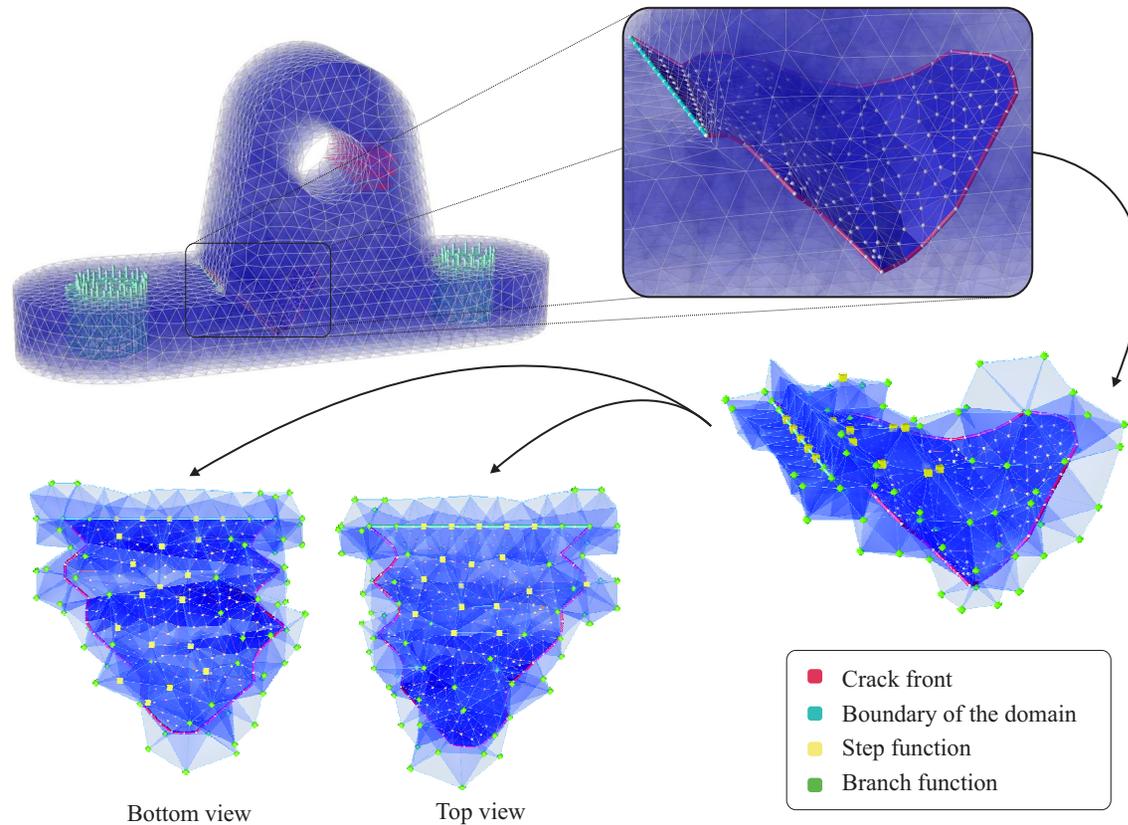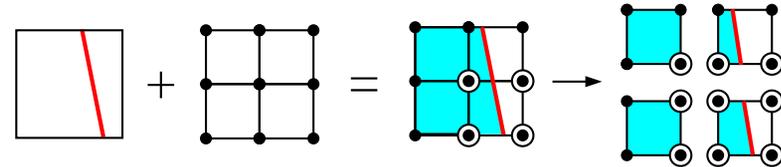
Bottom view   Top view

■ Crack front
■ Boundary of the domain
■ Step function
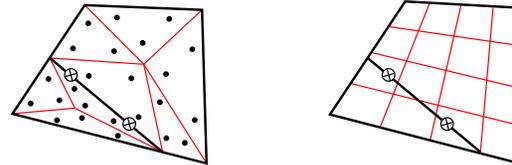■ Branch function

Finite Element Implementation

what's different compared to standard FEM?

▶ FEM procedures

    ▶ dealing with variable dofs/node

    ▶ integration of local K's
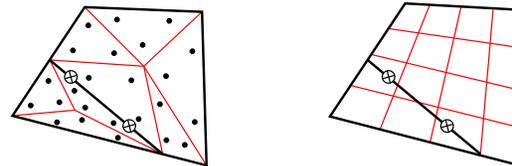
Finite Element Implementation

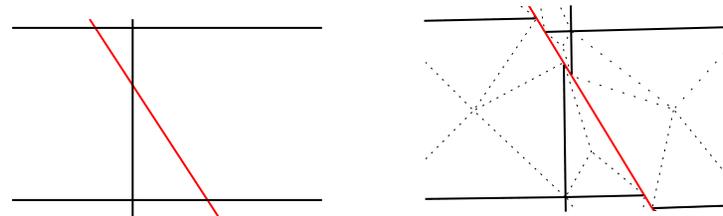what's different compared to standard FEM?

- ▶ **FEM procedures**
  - ▶ dealing with variable dofs/node

  - ▶ integration of local **K**'s

- ▶ **postprocessor**

Suggested Reading

► C. A. Duarte, O. N. Hamzeh, T. J. Liszka, and W. W. Tworzydlo. A generalized finite element method for the simulation of three-dimensional dynamic crack propagation. *Computer Methods in Applied Mechanics and Engineering*, 190:2227–2262, 2001.

► C.A. Duarte, L.G. Reno, and A. Simone. A high-order generalized FEM for through-thethickness branched cracks. *International Journal for Numerical Methods in Engineering*, 72(3):325–351, 2007. http://dx.doi.org/10.1002/nme.2012

► A. Simone. Partition of unity-based discontinuous finite elements: GFEM, PUFEM, XFEM. Revue Européenne de Génie Civil, 11(7-8):1045-1068, 2007. Special issue on "Damage and fracture in geomaterials" edited by M. Jirásek, D. Kondo and C. Viggiani.

► J.P. Pereira, C.A. Duarte, D. Guoy, and X. Jiao. Hp-Generalized FEM and crack surface representation for non-planar 3-D cracks. *International Journal for Numerical Methods in Engineering*, 77(5):601–633, 2009. http://dx.doi.org/10.1002/nme.2419

► J.P. Pereira, C.A. Duarte, and X. Jiao. Three-dimensional crack growth with hp-generalized finite element and face offsetting methods. *Computational Mechanics*, 46(3):431–453, 2010.

► J. Garzon C.A. Duarte and W. Buttlar. Analysis of reflective cracks in air field pavements using a 3-D generalized finite element method. *Road Materials and Pavement Design*, 11(2):459–477, 2010.